INTRODUCTION TO MEAN-FIELD SPIN GLASSES
EXERCISE SHEET 2

DAVID BELIUS

Don't worry if you don't have time to do all the exercises. All the models in the questions below are spherical models, that is the configuration space is the sphere $S_{N-1} = \{\sigma \in \mathbb{R}^N : |\sigma| = \sqrt{N}\}$, and the reference measure Q_N is uniform on S_{N-1}.

(1) For each N, let u denote an arbitrary vector in S_{N-1} and let $H_N(\sigma) = \sigma \cdot u$. Compute a variational formula for the limiting free energy of this model (the formula is the supremum of an explicit function of one variational parameter). Plot the formula for some different values of $\beta \geq 0$. Is there a phase transition?
Hint:
1. The limiting free energy is $\lim_{N\to\infty} \frac{1}{N} \log Q_N[\exp(\beta H_N(\sigma))]$.
2. Recall the entropy of the spherical integral: $Q_N[\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \geq \alpha\}] = \exp\left(\frac{N}{2}\log(1-\alpha^2) + o(N)\right)$

(2) Consider the spherical pure 1-spin model without (deterministic) external field. That is, the spherical model with Hamiltonian H_N given by a centered Gaussian process with covariance function $z(x) = x$. Prove that for any $\beta \geq 0$ and $h = 0$ the free energy $F_N(\beta, 0)$ converges in probability to a variational formula. Check numerically that the limiting free energy is strictly below the annealed free energy, i.e. that

$$\lim_{N\to\infty} F_N(\beta, 0) < \lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}[Z_N(\beta, 0) = \frac{\beta^2}{2} z(1) \quad \forall \beta > 0.$$

Hint:
1. H_N is the centered Gaussian process with covariance $\mathbb{E}[H_N(\sigma)H_N(\tau)] = Nz\left(\frac{\sigma\cdot\tau}{N}\right)$.
2. Use the case $p = 1$ of question (1) (b) from the first exercises sheet.

(3) Compute a variational formula for the limiting free energy like in Q1 for the same model, but with Hamiltonian $H_N(\sigma) = (\sigma \cdot u)^3$. Plot the formula for some different values of $\beta \geq 0$. Is there a phase transition?

(4) Let u, v denote orthogonal vectors in S_{N-1}. Compute a variational formula for the limiting free energy for the spherical model with Hamiltonian $H_N(\sigma) = (\sigma \cdot u)^2 (\sigma \cdot v)^2$ (the variational formula should have two variational parameters). Investigate the formula numerically. Is there a phase transition? If so, roughly for which value of β?