INTRODUCTION TO MEAN-FIELD SPIN GLASSES
EXERCISE SHEET 1

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Don't worry if you don't have time to do all the exercises. They are ordered in order of importance, so start from the beginning.

(1)

(a) Let $N \geq 1$, and let $J_{ij}, i, j = 1, \ldots N$ be independent standard Gaussian random variables. Let $H_N(\sigma) = \sum_{i,j=1}^{N} \frac{J_{ij}}{\sqrt{N}} \sigma_i \sigma_j$. Prove that for any $\sigma, \tau \in \mathbb{R}^N$ it holds that

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = Nz\left(\frac{\sigma \cdot \tau}{N}\right) \quad \text{for} \quad z(r) = r^2.$$

(b) Let $N \geq 1$, $p \geq 1$ and let $J_{i_1,\ldots,i_p}, i_1, \ldots, i_p = 1, \ldots, N$ be independent standard Gaussian random variables. Let κ be a constant to be determined. Let $H_N(\sigma) = \sum_{i,j=1}^{N} J_{i_1,\ldots,i_p} \sigma_{i_1} \ldots \sigma_{i_p}$. Prove that there is a κ such that for all $\sigma, \tau \in \mathbb{R}^N$

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = Nz\left(\frac{\sigma \cdot \tau}{N}\right) \quad \text{for} \quad z(r) = r^p.$$

(c) Let $N \geq 1, P \geq 1$ and $a_0, \ldots, a_P \geq 0$. Let $z(r) = \sum_{p=0}^{P} a_p r^p$. Let $J, J_i, J_{i_1 i_2}, \ldots, J_{i_1,\ldots,i_P}$ be i.i.d. standard Gaussian random variables. Prove that there exist constants $\kappa_0, \ldots, \kappa_p$ s.t. the covariance of $H_N(\sigma) = \kappa_0 J + \sum_{p=1}^{P} \kappa_p J_{i_1,\ldots,i_p}$ is

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = Nz\left(\frac{\sigma \cdot \tau}{N}\right).$$

(2) Let H_N be a mixed p-spin spin glass Hamiltonian with covariance function $z(r) = \sum_{p=0}^{\infty} a_p r^p, a_p \geq 0, z(1) < \infty$. Let Σ_N be $\{-1, 1\}^N$ or S_{N-1}, and let Q_N be the uniform measure on Σ_N. Let $Z_N(\beta)$ be the corresponding partition function.

(a) Prove that

$$\mathbb{E}[Z_N(\beta)] = \exp\left(N \frac{\beta^2}{2} z(1)\right) \quad \text{for all} \quad \beta \geq 0.$$

(b) Deduce that

$$\lim_{N \to \infty} \mathbb{P}\left[F_N(\beta) \geq \frac{\beta^2}{2} z(1) + \varepsilon\right] = 0 \text{ for all } \varepsilon > 0.$$

Remark: There are covariance functions z so that in fact $F_N(\beta) \to \frac{\beta^2}{2} z(1)$ in probability at high temperature.

(3) Consider the pure 0-spin Hamiltonian, i.e. a centered Gaussian process with covariance function $x(r) = 1 \forall r$. Prove that the limit $\lim_{N\to\infty} F_N(\beta)$ exists for all $\beta \geq 0$, and that

$$\lim_{N\to\infty} F_N(\beta) < \frac{\beta^2}{2} z(1).$$

(4) Let H_N be an arbitrary mixed-p Hamiltonian. Prove that

$$\mathbb{P}\left(\max_{\sigma\in\{-1,1\}^N} H_N(\sigma) \geq N\left(\sqrt{z(1) 2\log 2} + \varepsilon\right)\right) \to 0 \quad \text{for all} \quad \varepsilon > 0.$$