INTRODUCTION TO MEAN-FIELD SPIN GLASSES - *AND THE TAP APPROACH*

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For now these are "skeletal" notes, only providing the bare minimum definitions and results. The notes may be fleshed later.

1. Basic notation

The closed ball of radius r in $\mathbb{R}^N$ is denoted by

$$B_N(r) := \left\{\sigma \in \mathbb{R}^N : |\sigma| \le r\right\}. \tag{1}$$

The open ball of that radius is denoted by

$$B_N^\circ(r) := \left\{\sigma \in \mathbb{R}^N : |\sigma| < r\right\}. \tag{2}$$

The closed resp. open balls of radius $\sqrt{N}$ are denoted by

$$B_N := B_N(\sqrt{N}), \qquad B_N^\circ := B_N^\circ(\sqrt{N}). \tag{3}$$

The sphere of radius r embedded in $\mathbb{R}^N$ is denoted by

$$S_{N-1}(r) := \left\{\sigma \in \mathbb{R}^N : |\sigma| = r\right\}. \tag{4}$$

The sphere of radius $\sqrt{N}$ is denoted by

$$S_{N-1} := S_{N-1}(\sqrt{N}). \tag{5}$$

2. General framework of equilibrium statistical physics

(1) The configuration space: any set Σ.
(2) The reference measure: any probability measure Q on Σ.
(3) The Hamiltonian: any function $H : \Sigma \to \mathbb{R}$.
(4) The inverse temperature parameter: $\beta \ge 0$.
(5) The partition function for a given β: $Z(\beta) := Q\left[\exp(\beta H_N(\sigma))\right]$.
(6) The Gibbs measure for a given β: the measure G_β on Σ defined by

$$G_\beta(A) := Q\left[1_A \exp(\beta H_N(\sigma))\right], \quad A \subset \Sigma. \tag{6}$$

(In the above we ignore issues of measurability).

2.1. **Free energy.** Usually, we consider a sequence of configuration spaces Σ_N, reference measures Q_N, partition functions $Z_N(\beta)$ and Gibbs measure $G_{N,\beta}$, where the size of Σ_N grows with N (e.g. $\Sigma = \{-1,1\}^N, \Sigma = S_{N-1}$). In this case the quantity

$$F_N(\beta) := \frac{1}{N} \log Z_N(\beta) \tag{7}$$

is called the *free energy.*

3. Model definitions

Definition 1 (Hamiltonian of the Curie-Weiss model)**.** For any $N \geq 1$ the function $H_N : B_N \to \mathbb{R}$,

$$H_N(\sigma) := \sum_{i,j=1}^{N} \frac{1}{N} \sigma_i \sigma_j \tag{8}$$

is the Hamiltonian of the Curie-Weiss model.

The Hamiltonian of the Curie-Weiss model with an external field of strength $h \geq 0$ is defined as

$$H_N^h(\sigma) := H_N(\sigma) + h \sum_{i=1}^{N} \sigma_i. \tag{9}$$

Definition 2 (Ising Curie-Weiss model)**.** Let $\beta \geq 0, h \geq 0$, and $N \geq 1$. Let H_N^h be the Curie-Weiss Hamiltonian from (9). Let the configuration space be given by $\Sigma_N = \{-1,1\}^N$, and let the reference measure Q_N be the uniform measure $Q_N(A) = \frac{|A|}{|\Sigma|}, A \subset \Sigma_N$ on Σ_N. Using these, let the partition function, Gibbs measure and free energy be defined as in Section 2. These objects together constitute the Ising Curie-Weiss model with external field of strength h and inverse temperature β.

Definition 3 (Spherical Curie-Weiss model)**.** Let $\beta \geq 0, h \geq 0$, and $N \geq 1$. Let H_N^h be the Curie-Weiss Hamiltonian from (9). Let the configuration space be given by $\Sigma_N = S_{N-1}$, and let the reference measure Q_N be the uniform measure on Σ_N. Using these, let the partition function, Gibbs measure and free energy be defined as in Section . These objects together constitute the spherical Curie-Weiss model with external field of strength h and inverse temperature β.

Definition 4 (Sherrington-Kirkpatrick Hamiltonian)**.** Let $N \geq 1$. We have the following two alternative definitions.

(1) Let $J_{ij}, i, j = 1, \dots, N$ be i.i.d. standard Gaussian random variables, and $H_N : B_N \to \mathbb{R}$,

$$H_N(\sigma) = \sum_{i,j=1}^{N} \frac{J_{ij}}{\sqrt{N}} \sigma_i \sigma_j. \tag{10}$$

Then H_N is called the Sherrington-Kirkpatrick Hamiltonian, as is any random function with the same law as H_N.

(2) Let $H_N : B_N \to \mathbb{R}$ be a Gaussian process on B_N with zero mean everywhere, and covariance

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = N \left(\frac{\sigma \cdot \tau}{N} \right)^2. \tag{11}$$

Then H_N is called the Sherrington-Kirkpatrick Hamiltonian, as is any random function with the same law as H_N.

Definition 5 (Pure p-spin Hamiltonians)**.** Let $N \geq 1$ and $p \geq 1$. We have the following two alternative definitions.

(1) Let $J_{i_1,\dots,i_p}, i_1, \dots, i_p = 1, \dots, N$ be i.i.d. standard Gaussian random variables, and $H_N : B_N \to \mathbb{R}$,

$$H_N(\sigma) = \sum_{i,j=1}^{N} \frac{J_{ij}}{N^{\frac{p-1}{2}}} \sigma_{i_1} \dots \sigma_{i_p}. \tag{12}$$

Then H_N is called the pure p-spin Hamiltonian, as is any random function with the same law as H_N.

(2) Let $H_N : B_N \to \mathbb{R}$ be a Gaussian process on B_N with zero mean everywhere, and covariance

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = N \left(\frac{\sigma \cdot \tau}{N} \right)^p. \tag{13}$$

Then H_N is called the pure p-spin Hamiltonian, as is any random function with the same law as H_N.

Definition 6 (Mixed p-spin Hamiltonian)**.** Let $a_p \geq 0$ for $p = 0, 1, \dots$. Let $x(r) = \sum_{p=0}^{\infty} a_p r^p$. Assume $x(1) = \sum_{p=1}^{\infty} a_p < \infty$. Let $N \geq 1$. We have the following two alternative definitions.

(1) Let $H_N^p(\sigma), p \geq 0$, be independent pure p-spin Hamiltonians (as in Definition 5). Let

$$H_N(\sigma) = \sum_{p=0}^{\infty} \sqrt{a_p} H_N^p(\sigma). \tag{14}$$

Then H_N is called a mixed p-spin model with covariance function (or mixture) $x(r)$.

(2) Let $H_N : B_N \to \mathbb{R}$ be a Gaussian process on B_N with zero mean everywhere, and covariance

$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = Nx\left(\frac{\sigma \cdot \tau}{N}\right). \tag{15}$$

Then H_N is called a mixed p-spin Hamiltonian with covariance function (or mixture) $x(r)$, as is any random function with the same law as H_N.

4. Basic results

Lemma 7. *In each of the Definitions 4, 5, and also in Definition 6 provided $z(r)$ only has finitely many non-zero terms, the definition 1 implies the definition 2.*

Proof. Exercise sheet 1. □