Introduction to Mean-Field Spin Glasses - *and the TAP approach*

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Chapter 1

Introduction

1.1 Motivations

1.1.1 Motivation from physics

The first mean-field spin glass models originated in theoretical physics, in the study of magnetism.

1.1.1.1 *Spin* model: Ising model (1920)

The *Ising model* on a lattice is toy a model of magnetism. It consists of many ($N \to \infty$) individual "spin variables" which interact with each other.

The model is constructed by taking a box B_N of side-length N in d-dimensional lattice, and associating with each vertex $a \in B_N$ a spin variable $\sigma_a \in \{-1, 1\}$. The spins at all vertices of the box are collected in a configuration vector $\sigma \in \{-1, 1\}^{N^d}$. To each configuration vector one associates an energy via the *Hamiltonian*

$$H_N(\sigma) = \sum_{a \sim b} \sigma_a \sigma_b, \tag{1.1.1}$$

where the sum is over all *neighboring* vertices in the box B_N. Since

$$\sigma_a \sigma_b = \begin{cases} 1 & \text{if } \sigma_a = \sigma_b, \\ -1 & \text{if } \sigma_a \neq \sigma_b, \end{cases} \tag{1.1.2}$$

the energy $H_N(\sigma)$ is high if many neighboring spins align in the spin configuration $\sigma \in \{-1, 1\}^{N^d}$. The "interaction" is modelled by defining the Gibbs measure $G_{N,\beta}$ as the measure on $\{-1, 1\}^{N^d}$ proportional to $\exp(\beta H_N(\sigma))$. Here $\beta \geq 0$ is a parameter control the strength of the interaction.

The most important feature of the model is a *phase transition*. There is a critical β_c, separating the *high temperature* regime $\beta \in (0, \beta_c)$ from the *low temperature* regime $\beta > \beta_c$. At high temperature, a sample $\sigma \in \{-1, 1\}^{N^d}$ from $G_{N,\beta}$ for N large has roughly the same proportion of $+1$s and -1s, and the correlation between spins decays rapidly with distance - sufficiently distant spins are essentially independent with mean zero. At low temperature a sample $\sigma \in \{-1,1\}^{N^d}$ from $G_{N,\beta}$ for N large either has a significantly higher proportion of +1s or of −1s, with high probability. Moreover, spins remain correlated even as distance between spins tends to infinity. Taking $N \to \infty$ one can define a limiting Gibbs measure $G_{\infty,\beta}$. At low temperature that measure has *a pure state decomposition* of the form

$$G_{\infty,\beta} = \frac{1}{2} G_{\infty,\beta,+} + \frac{1}{2} G_{\infty,\beta,-}, \tag{1.1.3}$$

where $G_{\infty,\beta,\pm}$ are probability measures under which the spins variables have correlations that decay rapidly with distance, like the Gibbs measure itself does at at high temperature. However, the spatial average of the spins in a sample from $G_{\infty,\beta,\pm}$ will be close to a number $\pm m_*(\beta)$, where $m_*(\beta) > 0$ is called a *mean magnetization*. Under $G_{\infty,\beta,\pm}$, sufficiently distant spins are essentially independent with mean $\pm m_*(\beta)$. The meaning of the latter property is that $G_{\infty,\beta,\pm}$ are probability measures of a spin system that is *effectively at high temperature*. Thus the limiting Gibbs measure $G_{\infty,\beta}$ is either at high temperature if $\beta \in (0, \beta_c)$, or it is the combination of two probability measures at (effective) high temperature if $\beta > \beta_c$.

Note that (1.1.3) implies the following description of sampling from the limiting Gibbs measure $G_{\infty,\beta}$ at low temperature: first flip an unbiased coin to pick "+" or "−". Then sample a spin configuration from the corresponding $G_{\infty,\beta,\pm}$. Since each of $G_{\infty,\beta,\pm}$ have rapidly decaying correlations, all of the "global correlation" of $G_{\infty,\beta}$ at low temperature is "caused" by the coin flip.

1.1.1.2 *Spin glass* model: Edwards-Anderson model (1975)

There are exotic magnetic materials where the interactions between spins is *disordered*. This means that some pairs of spins interact in such a way as to cause them to align (like in the Ising model), while other pairs of spins interact in such a way as to cause them to take *opposite values*. The Edwards-Anderson model is obtained by modifying the Ising model to capture this behavior. Precisely, the Hamiltonian (1.1.1) is replaced by

$$H_N(\sigma) = \sum_{a \sim b} J_{ab} \sigma_a \sigma_b, \tag{1.1.4}$$

where J_{ab} are i.i.d. Gaussian random variables associated to each edge of the lattice. Now H_N is a random function, which makes the Gibbs measure $G_{N,\beta}$ a random probability measure. One is interested in the typical properties of (samples from) the Gibbs measure $G_{N,\beta}$, for a fixed typical realization of the $J_{a,b}$.

Given a realization of the $J_{a,b}$, the Gibbs measure is biased towards configurations where $\sigma_a = \sigma_b$ if $J_{a,b} > 0$ (as in the Ising model), but biased towards configurations where $\sigma_a = -\sigma_b$ if $J_{a,b} < 0$. Note that for a typical realization of the $J_{a,b}$ there will be some paths

$$a \sim b \sim c \sim d \sim a \tag{1.1.5}$$

of vertices in the lattice such that

$$J_{a,b} > 0 \quad J_{b,c} > 0 \quad J_{c,d} > 0 \quad J_{a,d} < 0. \tag{1.1.6}$$

This Gibbs measure will then on the one hand be biased towards $\sigma_a = \sigma_d$ because of the interactions on the path $a \sim b \sim c \sim d$, but on the other hand it will be biased towards $\sigma_a = -\sigma_d$ from the interaction along the edge $a \sim d$. The phenomenon of having contradicting biases built-in to the Gibbs measure is called *frustration*. Frustration makes the Edwards-Anderson model very difficult to study.

1.1.1.3 *Mean-field spin glass* model: SK model (1975)

The Sherrington-Kirkpatrick model is a mean-field version of the Edwards-Anderson model, that is a model that has been simplified by replacing the lattice with a complete graph. That is, one lets all pairs of spins interact rather than just neighbors. The Hamiltonian of the model with N spins is

$$H_N(\sigma) = \sum_{i,j=1}^{N} \frac{J_{a,b}}{\sqrt{N}} \sigma_a \sigma_b \tag{1.1.7}$$

for i.i.d. standard Gaussians $J_{a,b}$. Note that the varying sign of $J_{a,b}$ means that this model is still subject to frustration. The scaling factor $N^{-\frac{1}{2}}$ is needed to place the corresponding Gibbs measure $G_{N,\beta}$ into the "interesting regime", where a phase transition takes place at a finite $\beta \geq 0$. If a spin interacts with $N-1$ rather than $2d$ other spins, where $N \to \infty$, the individual reactions need to be scaled down so as to not "overwhelm" the system.

Early work by Thouless-Anderson-Palmer (TAP; 1977) suggested that at low temperature the SK model has a kind of pure state decomposition like (1.1.3), but vastly more complex. In particular, it appeared possible that the decomposition involves an *unbounded* number of pure states. Parisi's subsequent breakthrough (1979) confirmed this at the level of rigor of theoretical physics, with a method[1] quite far from the realm of arguments that can be turned into rigorous mathematics. His result took the form of a formula for the so called *free energy* of the model, whose form and derivation suggested an infinite hierarchy of pure states combined with random weights depending on the realization of the $J_{a,b}$. No statistical physics model with such a low temperature phase had ever been studied previously. Since then, clarifying and extending this picture has been a major effort in theoretical physics and mathematics, which is on-going.

Also on-going is the debate within theoretical physics if this kind of infinite pure state hierarchy is also present in the original lattice Edwards-Anderson model. The arguments for such a picture carrying over to the Edwards-Anderson model for very large d are stronger, but for the most physically relevant dimension $d = 3$ there is to this date no consensus in theoretical physics as to whether the low temperature phase is more similar to that of the SK model, or more similar to that of the Ising model.

1.1.2 Motivation from theoretical computer science

The k-SAT problem of theoretical computer science involves finding solutions to a *Boolean equation* in a large number of Boolean variables. The problem is *NP hard*, which means that most likely

[1] The *replica* method of theoretical physics combined with a replica symmetry breaking ansatz

there does not exist an algorithm that can solve any such Boolean equation efficiently. Theoretical computer scientists asked the natural question of if a typical *Boolean equation* can be efficiently solved. This lead them to formulate a model of a *random k-SAT formula*.

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1.1.3 Motivation from extreme value theory of correlated random fields

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1.2 Notation for balls and spheres

The closed ball of radius r in $\mathbb{R}^N$ is denoted by

$$B_N(r) := \left\{\sigma \in \mathbb{R}^N : |\sigma| \leq r\right\}. \tag{1.2.1}$$

The open ball of that radius is denoted by

$$B_N^\circ(r) := \left\{\sigma \in \mathbb{R}^N : |\sigma| < r\right\}. \tag{1.2.2}$$

The closed resp. open balls of radius $\sqrt{N}$ are denoted by

$$B_N := B_N(\sqrt{N}), \qquad B_N^\circ := B_N^\circ(\sqrt{N}). \tag{1.2.3}$$

The sphere of radius r embedded in $\mathbb{R}^N$ is denoted by

$$S_{N-1}(r) := \left\{\sigma \in \mathbb{R}^N : |\sigma| = r\right\}. \tag{1.2.4}$$

The sphere of radius $\sqrt{N}$ is denoted by

$$S_{N-1} := S_{N-1}(\sqrt{N}). \tag{1.2.5}$$

1.3 General framework of equilibrium statistical physics

An equilibrium statistical physics model defines a *Gibbs measure* on the space of all possible states (or *configurations*) of the system being modelled. A typical state of the entire system is modelled by a random sample from the Gibbs measure. To construct the Gibbs measure, one first defines a *Hamiltonian*, which assigns an energy to each possible state of the system, and uses it to exponentially tilt a "default measure" known as the *reference measure* (or *prior measure*) towards high values of the energy. The building blocks of such a model are the following.

1. The configuration space: any set Σ.
2. The reference measure: any probability measure Q on Σ.
3. The Hamiltonian: any function $H : \Sigma \to \mathbb{R}$.
4. The inverse temperature parameter: $\beta \geq 0$.
5. The partition function for a given β:
$$Z(\beta) := Q\left[\exp(\beta H(\sigma))\right]. \tag{1.3.1}$$
6. The Gibbs measure for a given β: the measure G_β on Σ defined by[2]
$$G_\beta(A) := Q\left[1_A \exp(\beta H(\sigma))\right], \quad A \subset \Sigma. \tag{1.3.2}$$

(In the above we ignore issues of measurability).

1.3.1 Free energy

Usually, we consider a sequence of configuration spaces Σ_N, reference measures Q_N, partition functions
$$Z_N(\beta) := Q_N\left[\exp(\beta H(\sigma))\right], \tag{1.3.3}$$
and Gibbs measure $G_{N,\beta}$, where the size of Σ_N grows with N (e.g. $\Sigma = \{-1,1\}^N$ or $\Sigma = S_{N-1}$). In this case the quantity
$$F_N(\beta) := \frac{1}{N}\log Z_N(\beta) \tag{1.3.4}$$
is called the *free energy*.

[2] In physics the convention is that a system is biased towards *low* rather than high energies. Correspondingly a physicist would replace $\exp(\beta H(\sigma))$ in (1.3.1)-(1.3.2) by $\exp(-\beta H(\sigma))$. The definition of Hamiltonians must then also be adapted to this convention. E.g. the Curie-Weiss Hamiltonian (2.1.1) would be replaced with $H_N(\sigma) = -\sum_{i,j=1}^N \frac{1}{N}\sigma_i\sigma_j$. Since the negative signs cancel out one clearly obtains exactly the same Gibbs measure regardless of the sign convention used. As is often the case in the mathematical literature, we use the convention without negative signs.

Chapter 2

Curie-Weiss model

2.1 Definition and first steps of free energy calculation

We first define the Curie-Weiss Hamiltonian.

Definition 2.1.1 (Hamiltonian of the Curie-Weiss model)**.** For any $N \geq 1$ the function $H_N : B_N \to \mathbb{R}$ given by

$$H_N(\sigma) := \sum_{i,j=1}^{N} \frac{1}{N} \sigma_i \sigma_j \tag{2.1.1}$$

is the *Curie-Weiss Hamiltonian*. The *Curie-Weiss Hamiltonian with an external field* of strength $h \geq 0$ is defined as

$$H_N^h(\sigma) := H_N(\sigma) + h \sum_{i=1}^{N} \sigma_i. \tag{2.1.2}$$

$\square$

Next we define the actual Curie-Weiss model(s), using the equilibrium statistical physics framework described in Section 1.3.

Definition 2.1.2 (Ising and Spherical Curie-Weiss model)**.** Let $\beta \geq 0, h \geq 0$, and $N \geq 1$. Let H_N^h be the Curie-Weiss Hamiltonian from (2.1.2). Let either

1. (*Ising* Curie-Weiss) the configuration space be given by $\Sigma_N = \{-1, 1\}^N$, and let the reference measure Q_N equal the uniform probability $Q_N^{\pm}$ on $\{-1, 1\}^N$, or

2. (*Spherical* Curie-Weiss) the configuration space be given by $\Sigma_N = S_{N-1}$, and let the reference measure Q_N equal the uniform probability Q_N^{sph} on S_{N-1}.

Based on these, let the partition function $Z_N(\beta, h) = Z_N(\beta)$, Gibbs measure $G_{N,\beta}{=}G_{N,\beta,h}$ and free energy $F_N(\beta) = F_N(\beta, h)$ be defined as in Section 1.3. These objects together constitute the *Ising Curie-Weiss model* resp. *spherical Curie-Weiss model*, with external field of strength h and inverse temperature β. $\square$

Our first goal is to compute the free energy of the Curie-Weiss model(s). They key for the computation is the identity

$$H_N(\sigma) = \sum_{i,j=1}^{N} \frac{1}{N}\sigma_i\sigma_j = N\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^2 = N\left(\frac{\sigma \cdot u}{N}\right)^2 \quad \forall \sigma \in S_{N-1} \tag{2.1.3}$$

satisfied by the Curie-Weiss Hamiltonian, where

$$u = (1, \dots, 1) \in S_{N-1}. \tag{2.1.4}$$

For the Curie-Weiss Hamiltonian with external field we obtain

$$H_N^h(\sigma) = H_N(\sigma) + Nh \times \frac{1}{N}\sum_{i=1}^{N}\sigma_i = Ng\left(\frac{\sigma \cdot u}{N}\right) \quad \forall \sigma \in S_{N-1}, \tag{2.1.5}$$

where

$$g(\alpha) = \alpha^2 + h\alpha, \quad \alpha \in (-1, 1). \tag{2.1.6}$$

Thus the partition function $Z_N(\beta, h)$ of the Ising or spherical Curie-Weiss models satisfies

$$Z_N(\beta, h) = Q_N\left[\exp\left(Ng\left(\frac{\sigma \cdot u}{N}\right)\right)\right] \quad \forall \beta, h \geq 0. \tag{2.1.7}$$

Since the integrand depends only on $\sigma \cdot u$ it makes sense to decompose the configuration space into "slices" along the direction u. These slices can be defined as

$$D_\alpha = \left\{\sigma \in S_{N-1} : \left|\frac{\sigma \cdot u}{N} - \alpha\right| \leq N^{-1/3}\right\}, \quad \alpha \in (-1, 1). \tag{2.1.8}$$

Letting

$$A = (N^{-1/3}\mathbb{Z}) \cap (-1, 1), \tag{2.1.9}$$

we then have

$$S_{N-1} = \bigcup_{\alpha \in A} D_\alpha. \tag{2.1.10}$$

Therefore

$$\begin{aligned} Q_N\left[\exp\left(Ng\left(\tfrac{\sigma \cdot u}{N}\right)\right)\right] &= \sum_{\alpha \in A} Q_N\left[1_{D_\alpha}\exp\left(Ng\left(\tfrac{\sigma \cdot u}{N}\right)\right)\right] \\ &= \sum_{\alpha \in A} Q_N\left[\exp\left(Ng\left(\tfrac{\sigma \cdot u}{N}\right)\right)|D_\alpha\right] \times Q_N[D_\alpha]. \end{aligned} \tag{2.1.11}$$

If g is say Lipschitz on $[-1, 1]$, then

$$Q_N\left[\exp\left(Ng\left(\frac{\sigma \cdot u}{N}\right)\right)|D_\alpha\right] = \exp\left(Ng(\alpha) + o(N)\right) \quad \text{uniformly over} \quad \alpha \in (-1, 1). \tag{2.1.12}$$

After using this in (2.1.11), it remains to estimate the "entropy" $Q_N[D_\alpha]$. This is the topic of the next section.

2.2 Entropy of spherical and Ising models

The following lemma is the standard large deviation tail estimate for the binomial distribution, in terms of the binary entropy function

$$I^{\pm} : [-1,1] \to [0,\infty), \qquad I^{\pm}(m) = \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m). \tag{2.2.1}$$

Let $Q_N^{\pm}$ denote the uniform probability on $\{-1,1\}^N$.

Lemma 2.2.1 (Entropy function for Ising models)**.** *Let*

$$u = (1,\dots,1) \in \mathbb{R}^N. \tag{2.2.2}$$

It holds that

$$Q_N^{\pm}\left[\left\{\sigma \in S_{N-1} : \tfrac{\sigma\cdot u}{N} \in [a,b]\right\}\right] = \exp\left(-NI^{\pm}(a) + o(N)\right) \text{ uniformly for } \begin{cases} 0 \le a < b \le 1, \\ |a-b| \ge 2N^{-1}. \end{cases} \tag{2.2.3}$$

A proof of Lemma 2.2.1 is given in the appendix (Part 4).
Let Q_N^{sph} denote the uniform probability on S_{N-1}. Let

$$I^{\mathrm{sph}} : (-1,1) \to [0,\infty), \qquad I^{\mathrm{sph}}(m) = -\frac{1}{2}\log(1-m^2). \tag{2.2.4}$$

Lemma 2.2.2 (Entropy function for spherical models)**.** *Fix an $\varepsilon > 0$. For any $u \in S_{N-1}$ it holds that*

$$Q_N^{\mathrm{sph}}\left[\left\{\sigma \in S_{N-1} : \frac{\sigma\cdot u}{N} \in [a,b]\right\}\right] = \exp\left(-NI^{\mathrm{sph}}(a) + o(N)\right) \text{ uniformly for } \begin{cases} 0 \le a < b \le 1-\varepsilon, \\ |a-b| \ge N^{-1/2}. \end{cases} \tag{2.2.5}$$

Remark 2.2.3. Heuristically, the formula 2.2.5 can be derived from the fact that the surface area of $S_{N-1}(r)$ is proportional to r^{N-1}. The argument is as follows: We have

$$S_{N-1}(r) = c_N r^{N-1}, \tag{2.2.6}$$

for a dimension dependent constant c_N which satisfies

$$\frac{c_{N-1}}{c_N} = \exp(o(N)), \tag{2.2.7}$$

(precisely, $c_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$, where the gamma function Γ satisfies $\Gamma(x+1) = x\Gamma(x)$ for $x > 1$).

The set $\left\{\sigma \in S_{N-1} : \frac{\sigma\cdot u}{N} = \alpha\right\}$ is a sphere of surface dimension $N-2$ embedded in $\mathbb{R}^N$, which has radius $\sqrt{N(1-\alpha^2)}$. Thus its $N-2$-dimensional surface area is

$$c_{N-1}\left(\sqrt{N(1-\alpha^2)}\right)^{N-2}. \tag{2.2.8}$$

For $\delta > 0$ the set

$$\left\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \in [\alpha, \alpha + \delta]\right\} \tag{2.2.9}$$

should then have roughly area

$$\delta \times c_{N-1} \left(\sqrt{N(1-\alpha^2)}\right)^{N-2}. \tag{2.2.10}$$

The surface area of S_{N-1} is $c_N(\sqrt{N})^{N-1}$. Thus $Q_N^{\mathrm{sph}}\left[\left\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \in [\alpha, \alpha+\delta]\right\}\right]$ should therefore be approximately equal to

$$\frac{c_{N-1}\left(\sqrt{N(1-\alpha^2)}\right)^{N-2}}{c_N(\sqrt{N})^{N-1}} = \exp(o(N)) \times (1-\alpha^2)^{\frac{N-2}{2}} = \exp\left(\frac{N}{2}\log(1-\alpha^2) + o(N)\right). \tag{2.2.11}$$

A proof of Lemma 2.2.2 is given in the appendix.

2.3 Curie-Weiss free energy formula from geometric decomposition

Proposition 2.3.1. *Fix $h \geq 0, \beta \geq 0$. Consider the free energy $F_N(\beta, h)$ of the Ising or spherical Curie-Weiss model, as defined in Definition 2.1.2. For the Ising Curie-Weiss model (i.e. if $Q_N = Q_N^{\pm}$) define $I = I^{\pm}$, and for the spherical Curie-Weiss mode, (i.e. if $Q_N = Q_N^{\mathrm{sph}}$) define $I = I^{\mathrm{sph}}$. In either case it holds that*

$$\lim_{N \to \infty} F_N(\beta, h) = \sup_{\alpha \in (-1,1)} F(\alpha), \tag{2.3.1}$$

where

$$F(\alpha) = \beta(\alpha^2 + h\alpha) - I(\alpha). \tag{2.3.2}$$

Recall the identity (2.1.6)-(2.1.7) for the partition function of the Curie-Weiss model (Ising and spherical). By the identity, Proposition 2.3.1 follows directly from the following slightly more general result.

Lemma 2.3.2. *Let $Q_N = Q_N^{\pm}$ and $I = I^{\pm}$, or $Q_N = Q_N^{\mathrm{sph}}$ and $I = I^{\mathrm{sph}}$. For any differentiable and Lipschitz continuous $g : [-1, 1] \to \mathbb{R}$ it holds that*

$$Z_{N,g} := Q_N \left[\exp\left(N g \left(\frac{\sigma \cdot u}{N} \right) \right) \right] = \exp\left(N \sup_{\alpha \in (-1,1)} F(\alpha) + o(N) \right), \tag{2.3.3}$$

where

$$F(\alpha) = g(\alpha) - I(\alpha). \tag{2.3.4}$$

The already discussed (2.1.8)-(2.1.12) are the beginning of the proof of Lemma (2.3.3). The result is

$$Z_{N,g} = \sum_{\alpha \in A} \underbrace{Q_N \left[\exp\left(N g \left(\frac{\sigma \cdot u}{N} \right) \right) | D_\alpha \right]}_{= \exp(Ng(\alpha) + o(N)) \text{ uniformly over } \alpha \in [-1,1]} \times Q_N[D_\alpha]. \tag{2.3.5}$$

It remains to estimate $Q_N[D_\alpha]$ using Lemma 2.2.1 resp. Lemma 2.2.2. Since $I(\alpha \pm N^{1/3}) = I(\alpha) + o(N)$ for $I \in \{I^{\pm}, I^{\mathrm{sph}}\}$for any fixed $\alpha \in (-1, 1)$, those lemmas imply that

$$Q_N[D_\alpha] = \exp(-N I(\alpha) + o(N)) \tag{2.3.6}$$

for any fixed $\alpha \in (N^{-1/3}, 1)$. By symmetry the estimate (2.3.6) also holds for $\alpha \in (-1, -N^{-1/3})$, and for $\alpha \in [-N^{-1/3}, N^{-1/3}]$ it is trivial since $I(\alpha) = o(1)$ for such α. Combining (2.3.5) and (2.3.6) and

$$|A| \leq 2N^{2/3} = \exp(o(N)) \tag{2.3.7}$$

”morally speaking” implies (2.3.3), modulo some minor technicalities. This is the essentially complete argument behind the proof of Lemma 2.3.2, and of Proposition 2.3.1.

Remark 2.3.3 (Gibbs measure decomposition). The method used to derive (2.3.3) is easily adapted to prove pure state decomposition for the Gibbs measure $G_{N,g}(A) = Q_N\left[1_A \exp\left(Ng\left(\frac{\sigma\cdot u}{N}\right)\right)\right]/Z_{N,g}$ corresponding to the partition function $Z_{N,g}$ in (2.3.3). For any α which is not within $o(1)$ of a global maximizer of $F(\alpha)$, the set D_α will have exponentially small measure under the $G_{N,g}$. Such an α will satisfy

$$F(\alpha) - \sup_{\alpha\in(-1,1)} F(\alpha) \leq -c \tag{2.3.8}$$

for a constant $c > 0$, so

$$G_{N,g}(D_\alpha) = \frac{Q_N\left[1_{D_\alpha} \exp\left(Ng\left(\frac{\sigma\cdot u}{N}\right)\right)\right]}{Z_{N,g}} = \frac{\exp\left(NF(\alpha) + o(N)\right)}{\exp\left(N\sup_{\alpha\in(-1,1)} F(\alpha) + o(N)\right)} \leq \exp\left(-cN + o(N)\right). \tag{2.3.9}$$

This suggests a Gibbs measure decomposition of the form

$$G_{N,g} \approx \frac{1}{\#\text{ glob. max of } F} \sum_{\alpha\in(-1,1):\alpha\text{ glob. max of } F} G_{N,g,\alpha}, \tag{2.3.10}$$

where $G_{N,g,\alpha}$ are "pure states" which are uniform probability measures on the set D_α. It is conceptually starlight-forward to make the heuristic approximation (2.3.10) quantitative and precise, using concrete estimates like (2.3.9).

The above discussion of pure state decompositions translates directly to the Curie-Weiss Gibbs measures $G_N(\beta, h)$ from Definition 2.1.2, since those $G_N(\beta, h)$ are in fact equal to $G_{N,g}$ for the g in (2.3.5)..

□

The rest of this section addresses the minor technical details of the fully rigorous proof of Lemma 2.3.2. If you are comfortable with the hand-wavy arguments above I recommend skipping ahead to Section 2.4, where we study the qualitative consequences of the variational formula (2.3.1).

2.3.1 Remaining technical details: ensuring uniformity in (2.3.6) and dealing with "edge" in (2.1.11)

It only remains to deal with the minor technicalities needed to make the argument for (2.3.3) fully rigorous.

Lemma 2.3.4. *For any Lipschitz g and $I \in \{I^{\pm}, I^{\mathrm{sph}}\}$ there is a $\varepsilon \in (0,1)$ such that*

$$\sup_{\alpha\in(-1,1):|\alpha|\geq 1-\varepsilon} \{g(\alpha) - I(1-\varepsilon)\} \leq \sup_{\alpha\in(-1,1)} F(\alpha) = \sup_{\alpha\in[-1+\varepsilon,1-\varepsilon]} F(\alpha), \tag{2.3.11}$$

where F is as in (2.3.4).

Proof. For both $I = I^{\pm}$ and $I = I^{\mathrm{sph}}$ the derivative $I'(\alpha)$ diverges as $\alpha \to \pm 1$, while $g'(\alpha)$ is bounded $\alpha \in [-1,1]$ since it is Lipschitz. Thus we can pick $\varepsilon \in (0,1)$ small enough so that $F'(-\alpha) > 0 > F'(\alpha)$ for $\alpha \in (-1,1)$ s.t. $|\alpha| \geq 1-\varepsilon$. This implies the equality in (2.3.11).

If $I = I^{\mathrm{sph}}$ it is obvious that the inequality in (2.3.11) holds for $\varepsilon \in (0,1)$ small enough, since $I^{\mathrm{sph}}(\alpha) \to -\infty$ as $\alpha \to \pm 1$. For $I = I^{\pm}$ the inequality holds for small enough $\varepsilon \in (0,1)$ since

$$\lim_{\varepsilon \to 0} \sup_{\alpha \in [1-\varepsilon,1)} \{g(\alpha) - I(1-\varepsilon)\} = g(1) - I(1) = F(1) \leq \sup_{\alpha \in (-1,1)} F(\alpha). \tag{2.3.12}$$

□

Proof of Lemma 2.3.2. By (2.3.5) we have

$$Z_{N,g} = \sum_{\alpha \in A} \exp(Ng(\alpha)) Q_N[D_\alpha] \tag{2.3.13}$$

For any $\varepsilon \in (0,1)$, Lemma 2.2.1 resp. Lemma 2.2.2 imply the bound

$$Q_N[D_\alpha] = \exp\left(-NI(\alpha) + o(N)\right) \quad \text{uniformly over} \quad \alpha \in [-1+\varepsilon, 1-\varepsilon]. \tag{2.3.14}$$

From this and (2.3.7) it follows that

$$\sum_{\alpha \in A \cap (-1+\varepsilon, 1-\varepsilon)} \exp(Ng(\alpha)) Q_N[D_\alpha] = \exp\left(N \sup_{\alpha \in (-1+\varepsilon, 1-\varepsilon)} F(\alpha) + o(N)\right). \tag{2.3.15}$$

Lemma 2.2.1 resp. Lemma 2.2.2 with $a = 1-\varepsilon$ and $b = 1$ we have for $\varepsilon \in (0,1)$

$$Q_N[D_\alpha] \leq \exp\left(-NI(1-\alpha) + o(N)\right) \quad \text{uniformly over} \quad \alpha \in (-1,1), |\alpha| \geq 1-\varepsilon. \tag{2.3.16}$$

Thus the "rest" can be bounded as

$$\sum_{\alpha \in A : |\alpha| \geq 1-\varepsilon} \exp(Ng(\alpha)) Q_N[D_\alpha] \leq \exp\left(N \sup_{\alpha \in (-1,1) : |\alpha| \geq 1-\varepsilon} \{g(\alpha) - I(1-\varepsilon)\} + o(N)\right). \tag{2.3.17}$$

Combining (2.3.13), (2.3.15), (2.3.17) and picking $\varepsilon \in (0,1)$ small enough so that (2.3.11) holds, we deduce (2.3.3).

□

2.4 Analysis of Curie-Weiss Free Energy formula

Important properties of the Ising resp. spherical Curie-Weiss model for different combinations of parameters $\beta \geq 0, h \geq 0$ can be deduced from the shape of the function

$$F(\alpha) = \beta(\alpha^2 + h\alpha) - I(\alpha) \tag{2.4.1}$$

from Proposition 2.3.1.

Example 2.4.1. (Curie-Weiss model $h = 0$) Consider the Ising and spherical Curie-Weiss models without external field ($h = 0$). The following figure shows plots of $F(\alpha)$ for different values of $\beta \geq 0$.

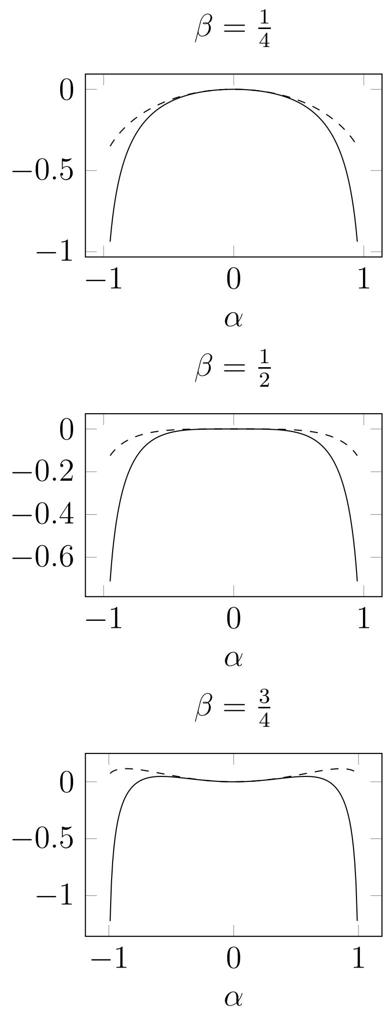


Figure 2.4.1: Plots of $F(\alpha)$ for $h = 0$, $I = I^{\text{sph}}$ (solid) and $I = I^{\pm}$ (dashed) and various values of β.

Letting

$$\beta_c := \frac{1}{\sqrt{2}} \tag{2.4.2}$$

we observe that both the Ising and spherical models satisfy

$$\beta \in [0, \beta_c] \implies F(\alpha) \text{ is uniquely maxmized at } \alpha = 0 \text{ and } F(0) = 0 \tag{2.4.3}$$

and

$$\beta > \beta_c \implies F(\alpha) \text{ is maximized at } \pm m_*(\beta), \text{ where } m_*(\beta) > 0 \text{ and } F(m_*(\beta)) > 0. \tag{2.4.4}$$

It is easy to prove (2.4.3) and (2.4.4) by using the series expansions

$$I^{\pm}(\alpha) = -\sum_{p \geq 2 : p \text{ even}} \frac{\alpha^p}{p(p-1)} \quad \text{and} \quad I^{\text{sph}}(\alpha) = -\frac{1}{2} \sum_{p \geq 2 : p \text{ even}} \frac{\alpha^p}{p}, \tag{2.4.5}$$

which both derive from the expansion of $\log(1+x)$.

The implications (2.4.3) and (2.4.4) exhibit a *phase transition* of the model. In (2.4.3)-(2.4.4) it is detected from the properties of the maximizer(s) of $F(\alpha)$. By Proposition 2.3.1 we have for $h = 0$ that

$$F_N(\beta) \to \begin{cases} 0 & \text{if } \beta \le \beta_c, \\ > 0 & \text{if } \beta > \beta_c, \end{cases} \tag{2.4.6}$$

which is another (related) expression of the phase transition. $h = 0$.

Recall the discussion of the pure state decomposition (1.1.3). The fact that $F(\alpha)$ has two global maximizers for $\beta > \beta_c$ suggests a pure state decomposition of the Curie-Weiss Gibbs measure of the form

$$G_{N,\beta} \approx \frac{1}{2} G_{N,-m_*(\beta)} + \frac{1}{2} G_{N,m_*(\beta)}, \tag{2.4.7}$$

where $G_{N,\pm m_*(\beta)}$ are roughly uniform on $D_{\pm m_*(\beta)} \cap \Sigma_N$. For the Ising Curie-Weiss model another way to describe such $G_{N,\pm m_*(\beta)}$ are as a probability measure where the spins $\sigma_i \in \{-1, 1\}$ are roughly i.i.d. with mean $\pm m_*(\beta)$.

Example 2.4.2. (Curie-Weiss model $h > 0$) For $h > 0$ the function $F(\alpha)$ always has a unique maximizer, and there is no phase transition in β. TBC.

Chapter 3

Mean-field spin glasses

3.1 Spin glass Hamiltonians and models

We now turn to actual mean-field *spin glass* models.

Definition 3.1.1 (Sherrington-Kirkpatrick (SK) Hamiltonian)**.** Let $N \geq 1$. We have the following two alternative definitions.

1. Let $J_{ij}, i, j = 1, \ldots, N$ be i.i.d. standard Gaussian random variables, and $H_N : B_N \to \mathbb{R}$,

$$H_N(\sigma) = \sum_{i,j=1}^{N} \frac{J_{ij}}{\sqrt{N}} \sigma_i \sigma_j. \tag{3.1.1}$$

   Then H_N is called the *Sherrington-Kirkpatrick (SK) Hamiltonian*, as is any random function with the same law as H_N.

2. Let $H_N : B_N \to \mathbb{R}$ be a Gaussian process on B_N with zero mean everywhere, and covariance

$$\mathbb{E}[H_N(\sigma) H_N(\tau)] = N \left(\frac{\sigma \cdot \tau}{N} \right)^2. \tag{3.1.2}$$

   Then H_N is called the *Sherrington-Kirkpatrick (SK) Hamiltonian*, as is any random function with the same law as H_N.

Remark 3.1.2 (Other conventions). Our convention for the the Curie-Weiss Hamiltonian (2.1.1) and the SK Hamiltonian (3.1.1) include the "self-interaction terms" $N^{-1}\sigma_i^2$ resp. $J_{ii}\sigma_i^2$. These are somewhat unnatural form the point of view of a physical spin model. Furthermore, our convention (2.1.1),(3.1.1) has two interaction terms for each pair unordered $\{i, j\}$. An alternative convention that removes these physically unnatural features is

$$H_N(\sigma) \overset{\text{C-W}}{=} \sum_{1 \leq i < j \leq N} \frac{1}{N} \sigma_i \sigma_j, \qquad H_N(\sigma) \overset{\text{SK}}{=} \sum_{1 \leq i < j \leq N} \frac{J_{ij}}{\sqrt{N}} \sigma_i \sigma_j. \tag{3.1.3}$$

The Gibbs measure obtained with these conventions is essentially the same as with (2.1.1), (3.1.1) with a modified ($\beta \to \frac{\beta}{2}$ for C-W and $\beta \to \frac{\beta}{\sqrt{2}}$ for SK). A disadvantage of the conventions (3.1.3) is that some formulas that are exact for the conventions (2.1.1), (3.1.1) become approximate. For the Curie-Weiss, the convention (2.1.1) has $H_N(\sigma) = N\left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right)^2$, while for (3.1.3) instead $H_N(\sigma) = \frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right)^2 - \frac{1}{N}\sum_{i=1}^N \sigma_i^2 = \frac{1}{2}\left(\frac{1}{N}\sum_{i=1}^N \sigma_i\right)^2 + O(1)$. For the convention (3.1.1) for the SK the covariance satisfies the identity (3.1.2), while in the convention (3.1.3) the covariance is $\mathbb{E}[H_N(\sigma)H_N(\tau)] = \frac{1}{2}N\left(\frac{\sigma\cdot\tau}{N}\right)^2 - \frac{1}{N}\sum_{i=1}^N \sigma_i^2 = \frac{1}{2}N\left(\frac{\sigma\cdot\tau}{N}\right)^2 + O(1)$. The exact identity (3.1.2) is especially convenient in the context of the generalized mixed p-spin Hamiltonians of Definition 3.1.5 below.

Definition 3.1.3 (Pure p-spin Hamiltonians)**.** Let $N \geq 1$ and $p \geq 0$. We have the following two alternative definitions.

1. Let $J_{i_1,\ldots,i_p}, i_1,\ldots,i_p = 1,\ldots,N$ be i.i.d. standard Gaussian random variables, and define $H_N : B_N \to \mathbb{R}$ by

$$H_N(\sigma) = \sum_{i_1,\ldots,i_p=1}^{N} \frac{J_{i_1,\ldots,i_p}}{N^{\frac{p-1}{2}}}\sigma_{i_1}\ldots\sigma_{i_p}. \tag{3.1.4}$$

   Then H_N is called the *pure p-spin Hamiltonian*, as is any random function with the same law as H_N.

2. Let $H_N : B_N \to \mathbb{R}$ be a Gaussian process on B_N with zero mean everywhere, and covariance

$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = N\left(\frac{\sigma\cdot\tau}{N}\right)^p. \tag{3.1.5}$$

   Then H_N is called the *pure p-spin Hamiltonian*, as is any random function with the same law as H_N.

Definition 3.1.4 (Covariance function)**.** A power-series $z(x) = \sum_{p\geq 0} a_p x^p$ with $a_p \geq 0$ and s.t. $z(x) < \infty$ for some $x \in (0,\infty)$ is called a *covariance function* or a *mixture*.

Definition 3.1.5 (Mixed p-spin Hamiltonian)**.** Let $z(x) = \sum_{p\geq 0} a_p x^p$ be a covariance function as defined in Definition 3.1.4. Assume $z(1) < \infty$. Let $N \geq 1$. We have the following two alternative definitions.

1. Let $H_N^p(\sigma), p \geq 0$, be independent pure p-spin Hamiltonians (as in Definition 3.1.3). Let

$$H_N(\sigma) = \sum_{p=0}^{\infty} \sqrt{a_p} H_N^p(\sigma). \tag{3.1.6}$$

   Then H_N is called a *mixed p-spin model with covariance function (or mixture) z(x)*, as is any random function with the same law as H_N.

2. Let $H_N : B_N \to \mathbb{R}$ be a Gaussian process on B_N with zero mean everywhere, and covariance
$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = Nz\left(\frac{\sigma \cdot \tau}{N}\right). \tag{3.1.7}$$
Then H_N is called a *mixed p-spin Hamiltonian with covariance function (or mixture)* $z(x)$, as is any random function with the same law as H_N.

Remark 3.1.6 (Regularity of H_N). It is obvious that the Hamiltonians defined in (3.1.1) and (3.1.4) are both a.s. finite and a.s. infinitely differentiable in all of $\mathbb{R}^N$, since they are polynomials in the spin variables. If a covariance function $z(x)$ only has finitely many non-zero terms and the H_N^p in (3.1.6) are constructed as in (3.1.4), then it is also obvious that the Hamiltonian (3.1.4) is a.s. finite and infinitely differentiable everywhere in $\mathbb{R}^N$. For $z(x)$ with infinitely many non-zero terms the same will be true provided the a_p decay fast enough, to ensure the appropriate convergence of the series (3.1.6). For instance exponentially decaying a_p is certainly enough for the corresponding Hamiltonian to be well-defined and a.s. finite and infinitely differentiable in $B_N(r)$ for some $r > 0$. However, in these notes we refrain from dealing with the required technicalities, and wherever it simplifies the proof we assume that z has finitely many non-zero terms.

Definition 3.1.7 (Mean-field spin glass Hamiltonian with external field)**.** Let H_N denote a mean-field spin glass Hamiltonian (the SK Hamiltonian as in Definition 3.1.1, a pure p-spin Hamiltonian as in Definition 3.1.3, or a mixed p-spin Hamiltonian like in Definition 3.1.5). The corresponding *SK* or *pure p-spin* or *mixed p-spin Hamiltonian* with an external field of strength $h \geq 0$ is defined as
$$H_N^h(\sigma) := H_N(\sigma) + h\sum_{i=1}^N \sigma_i, \quad \sigma \in B_N. \tag{3.1.8}$$
The *SK* or *pure p-spin* or *mixed p-spin Hamiltonian* Hamiltonian with external field in an arbitrary direction $u \in S_{N-1}$ is defined as
$$H_N^h(\sigma) := H_N(\sigma) + h(\sigma \cdot u), \quad \sigma \in B_N. \tag{3.1.9}$$

Remark 3.1.8 (Interpretation of $p = 0, 1$-components). By (3.1.4) with $p = 0$, the **pure** 0**-spin** Hamiltonian can be constructed as
$$H_N(\sigma) = \sqrt{N}J^0, \tag{3.1.10}$$
where J^0 is a standard normal random variable. It is thus a constant function.

By (3.1.4) with $p = 1$, the **pure** 1**-spin** Hamiltonian can be constructed as
$$H_N(\sigma) = \sigma \cdot J^1, \tag{3.1.11}$$
where J^1 is a vector with i.i.d. Gaussian entries. It is thus a linear function, taking the same form as the external field term in (3.1.9) for random h, u.

By (3.1.6), a **mixed** p**-spin Hamiltonian with** $a_0 > 0$ **or** $a_1 > 0$ can be constructed from a pure 0-spin Hamiltonian $\sigma \to \sqrt{N}J^0$, a pure 1-spin Hamiltonian $\sigma \cdot J^1$ and a mixed p-spin

Hamiltonian $\tilde{H}_N(\sigma)$ with covariance function $\tilde{z}(x) = \sum_{p\geq 2} a_p x^p$ with $a_0 = a_1 = 0$ - all mutually independent - by setting

$$H_N(\sigma) := \tilde{H}_N(\sigma) + \underbrace{\sqrt{a_1}\sigma \cdot J^1}_{\text{random ext. field}} + \underbrace{\sqrt{a_0}\sqrt{N}J^0}_{\text{random cons. shift}} . \tag{3.1.12}$$

In this formula $H_N(\sigma)$ can be interpreted as a Hamiltonian **with a random constant shift** (from the $p = 0$ component) and **a random external field** (from the $p = 1$ component).

3.2 Covariance of the Hamiltonians

Lemma 3.2.1. *In each of Definition 3.1.1, Definition 3.1.3, and also in Definition 3.1.5 provided $z(r)$ only has finitely many non-zero terms, the definition (1) in terms of interaction terms $J_{i_1 \dots i_p}$ implies the definition (2) in terms of the covariance.*

Proof. Let $p \geq 0$ and let $H_N(\sigma) = \sum_{i_1,\dots,i_p=1}^N N^{-\frac{p-1}{2}} J_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p}$ for $J_{i_1,\dots,i_p}$ i.i.d. standard Gaussians. Then for any $K \geq 1$ and $\sigma^1, \dots, \sigma^K \in B_N$ each $H_N(\sigma^i)$ is the linear combination of the same independent Gaussians, so $H_N(\sigma^1), \dots, H_N(\sigma^K)$ are jointly Gaussian. Furthermore

$$\mathbb{E}[H_N(\sigma)] = \sum_{i_1,\dots,i_p=1}^N N^{-\frac{p-1}{2}} \underbrace{\mathbb{E}[J_{i_1,\dots,i_p}]}_{=0} = 0. \tag{3.2.1}$$

For $\sigma, \tau \in B_N$ the covariance $\mathbb{E}[H_N(\sigma)H_N(\tau)]$ equals

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i_1,\dots,i_p=1}^N N^{-\frac{p-1}{2}} J_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p}\right)\left(\sum_{j_1,\dots,j_p=1}^N N^{-\frac{p-1}{2}} J_{j_1,\dots,j_p} \tau_{j_1} \dots \tau_{j_p}\right)\right] \\
&= \mathbb{E}\left[\sum_{i_1,\dots,i_p=1}^N \sum_{j_1,\dots,j_p=1}^N N^{-(p-1)} J_{i_1,\dots,i_p} J_{j_1,\dots,j_p} \sigma_{i_1} \dots \sigma_{i_p} \tau_{j_1} \dots \tau_{j_p}\right] \\
&= N^{-(p-1)} \sum_{i_1,\dots,i_p=1}^N \sum_{j_1,\dots,j_p=1}^N \underbrace{\mathbb{E}\left[J_{i_1,\dots,i_p} J_{j_1,\dots,j_p}\right]}_{=\begin{cases} 1 & \text{if } (i_1,\dots,i_p) = (j_1,\dots,j_p) \\ 0 & \text{otherwise} \end{cases}} \sigma_{i_1} \dots \sigma_{i_p} \tau_{j_1} \dots \tau_{j_p} \\
&= N^{-(p-1)} \sum_{i_1,\dots,i_p=1}^N \sigma_{i_1} \dots \sigma_{i_p} \tau_{i_1} \dots \tau_{i_p} \\
&= N^{-(p-1)} (\sigma \cdot \tau)^p \\
&= N \left(\tfrac{\sigma \cdot \tau}{N}\right)^p \checkmark
\end{aligned} \tag{3.2.2}$$

□

3.3 Mixed p-spin Hamiltonians as isotropic random fields

An *isotropic random field* is a random field with a rotational invariant distribution. The next lemma states that a centered (i.e. mean zero) Gaussian random field on the sphere S_{N-1} has a rotationally invariant law iff its covariance for a pair of points on the sphere depends only on the inner product. This is precisely the form of the covariance of the mixed p-spin Hamiltonians.

Lemma 3.3.1. *If $N \geq 1$, $r > 0$ and H_N is a centered Gaussian random field on $S_{N-1}(r)$ s.t.*

$$(H_N(\sigma))_{\sigma \in B_N(r)} \overset{law}{=} (H_N(O\sigma))_{\sigma \in S_{N-1}(r)} \text{ for all orthogonal } O \in \mathbb{R}^{N\times N}, \tag{3.3.1}$$

then the covariance of $H_N(\sigma)$ takes the form

$$\mathbb{E}[H_N(\sigma)H_N(\tau)] = z(\sigma \cdot \tau), \quad \sigma, \tau \in S_{N-1}(r), \tag{3.3.2}$$

for some function $z : [0, r] \to \mathbb{R}$.

It is natural to ask for which functions z the map $(\sigma, \tau) \to z(\sigma \cdot \tau)$ is a well-defined covariance function, i.e. is a positive semi-definite function. The next theorem gives a partial answer.

Theorem 3.3.2 (Schoenberg's theorem)**.** *If $r > 0$ and $z : [0, r] \to \mathbb{R}$ is a function such that for each $N \geq 1$ the map*

$$(\sigma, \tau) \to z(\sigma \cdot \tau) \quad \text{from} \quad B_N(r) \times B_N(r) \quad \text{to} \quad \mathbb{R} \tag{3.3.3}$$

is positive semi-definite, then z is a covariance function in the sense of Definition 3.1.4.

Lemma 3.3.1 and Theorem 3.3.2 imply that the class of mixed p-spin Hamiltonians consists of *essentially all* rotationally invariant Gaussian random fields on the high-dimensional sphere. The only such Gaussian random fields that are not mixed p-spin Hamiltonians in the sense of Definition 3.1.5 are those with a covariance of the form (3.3.2) for a z such that (3.3.3) is positive semi-definite only for some N.

3.4 Annealed free energy and free energy upper bound

Since it is a $N \to \infty$-dimensional integral, computing the partition function $Z_N(\beta)$ is in general a difficult task. But for mixed p-spin models without external fields the expectation $\mathbb{E}[Z_N(\beta)]$ takes a simple form.

Lemma 3.4.1 (Annealed FE for $h = 0$)**.** *Let H_N be a mixed p-spin Hamiltonian with arbitrary covariance function $z(x)$ such that $z(1) < \infty$, and without (deterministic) external field (i.e. $h = 0$). Let Σ_N be any subset of S_{N-1}, and Q_N be any probability measure on Σ_N. Let $Z_N(\beta)$ be the corresponding partition function. Then*

$$\mathbb{E}[Z_N(\beta)] = \exp\left(\frac{\beta^2}{2} z(1) N\right) \quad \textit{for all} \quad \beta \geq 0. \tag{3.4.1}$$

Proof. Note that

$$\mathbb{E}[Z_N(\beta)] \overset{(1.3.3)}{=} \mathbb{E}[Q_N[\exp(\beta H_N(\sigma))]] \overset{\text{Fubini}}{=} Q_N[\mathbb{E}[\exp(\beta H_N(\sigma))]]. \tag{3.4.2}$$

Recall from Definition 3.1.5 that for every σ the random variable $H_N(\sigma)$ is a centered Gaussian with variance $\mathbb{E}[H_N(\sigma)^2] = N z(1)$.

$$\begin{gathered}\text{For a centered Gaussian r.v. } A \text{ with variance } s^2 \\ \text{the exponential moment equals } \mathbb{E}[\exp(\lambda A)] = \exp\left(\tfrac{\lambda^2}{2} s^2\right).\end{gathered} \tag{3.4.3}$$

Thus

$$\mathbb{E}[\exp(\beta H_N(\sigma))] = \exp\left(\frac{\beta^2}{2} z(1) N\right) \quad \text{for all} \quad \sigma \in S_{N-1}. \tag{3.4.4}$$

Since Q_N is assumed to be supported on S_{N-1}, the right-most expression in (3.4.2) thus equals the l.h.s. of (3.4.1). $\square$

Lemma 3.4.1 provides an upper bound for the free energy $F_N(\beta)$ valid for any $\beta \geq 0$ (when $h = 0$). The next two corollaries give to variants of this bound.

Corollary 3.4.2. *For $z(x), h = 0, H_N, \Sigma_N, Q_N$ as in Lemma 3.4.1*

$$\mathbb{E}[F_N(\beta)] \leq \frac{\beta^2}{2} z(1). \tag{3.4.5}$$

Proof. Since log is concave Jensen's inequality implies that

$$\mathbb{E}[F_N(\beta)] \overset{(1.3.4)}{=} \mathbb{E}\left[\frac{1}{N}\log(Z_N(\beta))\right] \overset{\text{Jensen}}{\leq} \frac{1}{N}\log \mathbb{E}[Z_N(\beta)] \overset{(3.4.1)}{=} \frac{\beta^2}{2} z(1). \tag{3.4.6}$$

$\square$

Corollary 3.4.3. *For $z(x), h = 0, H_N, \Sigma_N, Q_N$ as in Lemma 3.4.1*

$$\mathbb{P}\left(F_N(\beta) \geq \frac{\beta^2}{2}z(1) + \varepsilon\right) \leq e^{-\varepsilon N} \text{ for all } \varepsilon > 0, N \geq 1. \tag{3.4.7}$$

Proof. Using Markov's inequality:

$$\begin{aligned}\mathbb{P}\left(F_N(\beta) \geq \tfrac{\beta^2}{2}z(1) + \varepsilon\right) &\overset{(1.3.4)}{=} \mathbb{P}\left(Z_N(\beta) \geq \exp\left(N\tfrac{\beta^2}{2}z(1) + \varepsilon N\right)\right) \\ &\overset{\text{Markov}}{\leq} \frac{\mathbb{E}[Z_N(\beta)]}{\exp\left(N\frac{\beta^2}{2}z(1)\right) \times \exp(\varepsilon N)} \\ &\overset{(3.4.1)}{=} \frac{1}{\exp(\varepsilon N)}.\end{aligned} \tag{3.4.8}$$

□

The quantity$\frac{\beta^2}{2}z(1)$ coming from taking the expectation of $Z_N(\beta)$ is called the *annealed free energy*. The actual free energy $F_N(\beta)$ (or its expectation $\mathbb{E}[F_N(\beta)]$) is called the *quenched free energy*. As we will see later in Section 3.6, the upper bound (3.4.7) is in fact tight to leading order if $h = a_0 = a_1 = 0$ and β is small enough. In fact, for $z(x)$ such that $a_0 = a_1 = 0$, the main phase transition β_c is defined as the β_c up to which $F_N(\beta)$ converges to the annealed free energy. holds.

Definition 3.4.4 (Critical β_c for $h = a_0 = a_1 = 0$)**.** For $z(x)$ such that $a_0 = a_1 = 0$ and $Q_N = Q_N^{\pm}$ or $Q_N = Q_N^{\text{sph}}$ let

$$\beta_c = \beta_c(z, Q_N) := \sup\left\{\beta \in [0,\infty) : F_N(\beta) \xrightarrow{\mathbb{P}} \frac{\beta^2}{2}z(1)\right\}. \tag{3.4.9}$$

In Section 3.6 we will prove that $\beta_c > 0$ for all $h = a_0 = a_1 = 0$.

The next example gives an example of a covariance function $z(x)$ such that the upper bounds (3.4.5), (3.4.7) are not tight for any $\beta \geq 0$.

Example 3.4.5 (Quenched free energy not equal to annealed free energy for pure 0-spin Hamiltonian)**.** For the mixed p-spin model with covariance $z(x) = 1$, $h = 0$ and any reference measure Q_N on S_{N-1}, the free energy $F_N(\beta)$ satisfies

$$\lim_{N\to\infty} F_N(\beta) \to 0 \quad \text{for all} \quad \beta > 0. \tag{3.4.10}$$

Thus

$$\lim_{N\to\infty} F_N(\beta) < \frac{\beta^2}{2}z(1) \quad \text{for all} \quad \beta > 0, \tag{3.4.11}$$

where the equality is *strict*.

To verify (3.4.10), construct the Hamiltonian as $H_N(\sigma) = \sqrt{N}J$ for a standard Gaussian J (recall (3.1.10)), and note that by a standard Gaussian tail bound

$$\mathbb{P}(|J| \geq N^{1/4}) \leq 2\exp\left(-\frac{(N^{1/4})^2}{2}\right) \to 0 \quad \text{as} \quad N \to \infty. \tag{3.4.12}$$

On the event $\{|J| \leq N^{1/4}\}$

$$Z_N(\beta) = Q_N[\exp(\beta\sqrt{N}J)] = \exp(\beta\sqrt{N}J) = \exp(o(N)), \tag{3.4.13}$$

which implies

$$F_N(\beta) = o(1). \tag{3.4.14}$$

Example 3.4.5 is quite trivial, since for $z(x) = 1$ the Hamiltonian is constant in σ. But in fact, the annealed upper bound is loose not only in this case, but for any mixed p-spin model where at least one h, a_0, a_1 is positive (i.e. for any model with a deterministic or random external field, or a constant random shift).

3.5 Second moment free energy lower bound

In Section 3.4 we proved that in the absence of external field ($h = 0$), the annealed free energy $\frac{\beta^2}{2}z(1)$ is always an upper bound for the quenched free energy $F_N(\beta)$. In this section we give a sufficient condition (namely (3.5.1)) for the quenched free energy to actually equal the annealed free energy, i.e. for $F_N(\beta) = \frac{\beta^2}{2}z(1) + o(1)$.

Proposition 3.5.1. *Let $P \geq 2$ and let $z(x) = \sum_{p=0}^{P} a_p x^p$ be a covariance function. For each $N \geq 1$, let H_N be a mixed p-spin Hamiltonian with covariance function z and without external field ($h = 0$). Let $Q_N = Q_N^{\pm}$ and $I = I^{\pm}$ (Ising model), or $Q_N = Q_N^{\text{sph}}$ and $I = I^{\text{sph}}$ (spherical model), and let $F_N(\beta)$ denote the corresponding free energy. If $\beta \geq 0$ and $z(x)$ are such that*

$$\sup_{\alpha\in(-1,1)} \left\{\beta^2 z(\alpha) - I(\alpha)\right\} \leq 0, \tag{3.5.1}$$

then

$$\lim_{N\to\infty} \mathbb{P}\left(F_N(\beta) \geq \frac{\beta^2}{2}z(1) - \varepsilon\right) = 1 \quad \textit{for all} \quad \varepsilon > 0. \tag{3.5.2}$$

Remark 3.5.2.

1. While stated for $z(x) = \sum_{p=0}^{P} a_p x^p$, the Proposition is in practice only useful if $a_0 = a_1 = 0$ - otherwise the condition (3.5.1) is never satisfied for any $\beta \geq 0$.

2. If $a_0 = 0$ then $\beta^2 z(0) - I(0) = 0$, cf. (3.5.1).

3. Proposition 3.5.1 assumes $z(x)$ has only finitely many non-zero terms. This is a technical condition to shorten the proof - it can easily be weakened to allow infinite sequences a_p with fast enough decay.

Combining Proposition 3.5.1 with Corollary 3.4.3 shows that if $h = 0$ and (3.5.1) holds, then the quenched free energy $F_N(\beta)$ in fact converges to the annealed free energy in probability.

Corollary 3.5.3. *Let $z(x), h = 0, H_N, Q_N$ be as in the statement of Proposition 3.5.1. If (3.5.1) holds then*

$$F_N(\beta) \xrightarrow{\mathbb{P}} \frac{\beta^2}{2}z(1), \quad \textit{as} \quad N \to \infty. \tag{3.5.3}$$

The proof of Proposition 3.5.1 uses the *second moment method* to show concentration of $Z_N(\beta)$ around its expectation. As such, it bounds the second moment $\mathbb{E}[Z_N(\beta)^2]$ from above, and then uses the *Paley-Zygmund inequality*. The Paley-Zygmund inequality states that

$$\begin{gathered}\mathbb{P}(X \geq \theta\mathbb{E}[X]) \geq (1-\theta)^2 \tfrac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]} \quad \text{for} \quad \theta \in [0,1],\\ \text{for any r.v. } X \text{ s.t. } X \overset{\text{a.s.}}{\geq} 0 \text{ and } \mathbb{E}[X] < \infty.\end{gathered} \tag{3.5.4}$$

Applied to $Z_N(\beta)$ and using Lemma 3.4.1 this gives

$$\mathbb{P}\left(Z_N(\beta) \geq \theta \exp\left(N\frac{\beta^2}{2}z(1)\right)\right) \geq (1-\theta)^2 \frac{\exp(\beta^2 z(1))}{\mathbb{E}[Z_N(\beta)^2]}. \tag{3.5.5}$$

The following lemma gives a simple formula for the second moment $\mathbb{E}[Z_N(\beta)^2]$.

Lemma 3.5.4. *Let $z(x), h = 0, H_N$ be as in the statement of Proposition 3.5.1. For any reference probability measure Q_N*

$$\mathbb{E}[Z_N(\beta)^2] = \mathbb{E}[Z_N(\beta)]^2 \times (Q_N \times Q_N) \left[\exp\left(N\beta^2 z\left(\frac{\sigma \cdot \tau}{N}\right)\right)\right], \tag{3.5.6}$$

where under the probability measure $Q_N \times Q_N$ the r.v.s. σ, τ are independent with law Q_N.

Proof. We can write

$$Z_N(\beta)^2 = Q_N\left[\exp\left(\beta H_N(\sigma)\right)\right]^2 = Q_N\left[\exp\left(\beta H_N(\sigma)\right)\right] Q_N\left[\exp\left(\beta H_N(\tau)\right)\right], \tag{3.5.7}$$

where in the second integral on the r.h.s. we integrate over the variable τ rather than σ. By Fubini's theorem the r.h.s. equals

$$(Q_N \times Q_N)\left[\exp\left(\beta H_N(\sigma)\right)\exp\left(\beta H_N(\tau)\right)\right] = (Q_N \times Q_N)\left[\exp\left(\beta\left\{H_N(\sigma) + H_N(\tau)\right\}\right)\right], \tag{3.5.8}$$

for $Q_N \times Q_N$ as described in the statement. Using (3.5.7) and (3.5.8) and Fubini's theorem again we obtain

$$\mathbb{E}[Z_N(\beta)^2] = (Q_N \times Q_N)\left[\mathbb{E}\left[\exp\left(\beta\left\{H_N(\sigma) + H_N(\tau)\right\}\right)\right]\right]. \tag{3.5.9}$$

Now for any fixed σ, τ the random variables $H_N(\sigma), H_N(\tau)$ are jointly Gaussian with mean zero, variance $Nz(1)$ and covariance $Nz\left(\frac{\sigma \cdot \tau}{N}\right)$. This implies that $H_N(\sigma) + H_N(\tau)$ is Gaussian with mean zero and variance

$$\mathbb{E}\left[(H_N(\sigma) + H_N(\tau))^2\right] = Nz(1) + N2z\left(\frac{\sigma \cdot \tau}{N}\right) + Nz(1) = N2\left(z(1) + z\left(\frac{\sigma \cdot \tau}{N}\right)\right). \tag{3.5.10}$$

Thus

$$\mathbb{E}\left[\exp\left(\beta\left\{H_N(\sigma) + H_N(\tau)\right\}\right)\right] \overset{(3.4.3)}{=} \exp\left(N\beta^2\left\{z(1) + z\left(\frac{\sigma \cdot \tau}{N}\right)\right\}\right) \quad \text{for all} \quad \sigma, \tau. \tag{3.5.11}$$

Using this in (3.5.9) proves (3.5.6). $\square$

By symmetry of the distributions $Q_N = Q_N^{\pm}$ and $Q_N = Q_N^{\mathrm{sph}}$, the integral over two spin vectors σ, τ in (3.5.6) can be reduced to an integral over only σ.

Corollary 3.5.5. *For $h = 0$, all $\beta \geq 0$ and $Q_N = Q_N^{\pm}$ or $Q_N = Q_N^{\mathrm{sph}}$*

$$\mathbb{E}[Z_N(\beta)^2] = \mathbb{E}[Z_N(\beta)]^2 \times Q_N\left[\exp\left(N\beta^2 z\left(\frac{\sigma \cdot u}{N}\right)\right)\right], \tag{3.5.12}$$

where

$$u = (1, \ldots 1) \in S_{N-1}. \tag{3.5.13}$$

Proof. Consider $\sigma \cdot \tau$ in (3.5.6) when $Q_N = Q_N^{\pm}$. For each fixed $\tau \in \{-1, 1\}^N$, the $Q_N^{\pm}$-law of $\sigma \cdot \tau = \sum_{i=1}^N \sigma_i \tau_i$ coincides with the $Q_N^{\pm}$-law of $\sigma \cdot u = \sum_{i=1}^N \sigma_i$. Thus when $Q_N = Q_N^{\pm}$ the identity (3.5.12) holds. When $Q_N = Q_N^{\mathrm{sph}}$, the Q_N^{sph} -law of $\sigma \cdot \tau$ for fixed $\tau \in S_{N-1}$ is independent of τ. Thus also for $Q_N = Q_N^{\mathrm{sph}}$ the identity (3.5.12) follows from (3.5.6). $\square$

The next step is to bound the r.h.s. of (3.5.12) in terms of the supremum in (3.5.1). It is natural to decompose the integral using the sets

$$D_\alpha = \left\{ \sigma \in S_{N-1} : \left| \frac{\sigma \cdot u}{N} - \alpha \right| \leq N^{-\frac{1}{3}} \right\}, \quad \alpha \in (-1, 1), \tag{3.5.14}$$

like we did to analyze the Curie-Weiss model in Section 2.3. But before repeating the argument, we should note that

$$Q_N \left[\exp \left(N \beta^2 z \left(\frac{\sigma \cdot u}{N} \right) \right) \right] \tag{3.5.15}$$

on the r.h.s. of (3.5.12) is *exactly* of the form of the l.h.s. of (2.3.3) with

$$g(\alpha) = \beta^2 z(\alpha). \tag{3.5.16}$$

Furthermore g is Lipschitz under the assumption on z of Proposition 3.5.1. Thus simply applying Lemma 2.3.2 allows us to deduce that

$$Q_N \left[\exp \left(N \beta^2 z \left(\frac{\sigma \cdot u}{N} \right) \right) \right] = \exp \left(N \sup_{\alpha \in (-1,1)} \left\{ \beta^2 z(\alpha) - I(\alpha) \right\} + o(N) \right). \tag{3.5.17}$$

Thus we have proved the following

Lemma 3.5.6. *Let $z(x), h = 0, H_N$ be as in Proposition 3.5.1. Let $Q_N = Q_N^{\pm}$ and $I = I^{\pm}$, or $Q_N = Q_N^{\text{sph}}$ and $I = I^{\text{sph}}$. For $h = 0$ and all $\beta \geq 0$*

$$\mathbb{E}[Z_N(\beta)^2] \leq \mathbb{E}[Z_N(\beta)]^2 \times \exp \left(N \sup_{\alpha \in (-1,1)} \left\{ \beta^2 z(\alpha) - I(\alpha) \right\} + o(N) \right). \tag{3.5.18}$$

By the previous lemma, the condition (3.5.1) implies that

$$\mathbb{E}[Z_N(\beta)^2] \leq \mathbb{E}[Z_N(\beta)]^2 \times \exp(o(N)). \tag{3.5.19}$$

Plugging this into (3.5.5) with say $\theta = \frac{1}{2}$ yields

$$\mathbb{P} \left(Z_N(\beta) \geq \frac{1}{2} \underbrace{\exp \left(N \frac{\beta^2}{2} z(1) \right)}_{= \mathbb{E}[Z_N(\beta)]^2} \right) \geq \exp(-o(N)), \tag{3.5.20}$$

which implies that

$$\mathbb{P} \left(F_N(\beta) \geq \frac{\beta^2}{2} z(1) - c N^{-1} \right) \geq \exp(-o(N)), \tag{3.5.21}$$

(for $c = \log 2^{-1}$). The inequality (3.5.21) states that there is a sequence a_N s.t. the probability on l.h.s. exceeds $\exp(-a_N)$ and $\lim_{N \to \infty} \frac{a_N}{N} = 0$. Comparing (3.5.21) to our goal (3.5.2), we see that (3.5.21) falls short. Indeed, our actual goal (3.5.2) states that the probability on the l.h.s. of

(3.5.21) tends to one, while (3.5.21) only gives the weaker statement that this probability is *not* exponentially small in N. For instance it doesn't rule out that the probability tends to zero at rate $O(N^{-100})$.

The most direct way to improve the lower bounds in (3.5.20)-(3.5.21) is to bound the second moment more precisely. In particular, *if* one can show the asymptotic

$$\mathbb{E}[Z_N(\beta)^2] = (1+o(1))\mathbb{E}[Z_N(\beta)]^2, \tag{3.5.22}$$

which is a stronger estimate than (3.5.19), then the Paley-Zygmund inequality (3.5.5) with say $\theta = \exp(-\sqrt{N})$ implies that

$$\lim_{N\to\infty} \mathbb{P}\left(Z_N(\beta) \geq \exp\left(N\frac{\beta^2}{2}z(1) - \sqrt{N}\right)\right) = 1, \tag{3.5.23}$$

which in turns would imply (3.5.2).

The asymptotic (3.5.22) does in fact holds if $a_0 = a_1 = a_2 = 0$ and the sup in (3.5.18) is uniquely maximized at $\alpha = 0$. This can be proved by estimating the r.h.s. of (3.5.6) by an integral of the form $\int \exp(Nf(\alpha))d\alpha$ for $f(\alpha)$ uniquely maximized at $\alpha = 0$. Such an integral can be estimated using Laplace's method. The leading order estimate of Laplace's method then yields (3.5.18), while the subleading correction to the Laplace's method yields (3.5.22). However, if $a_2 > 0$ and the sup in (3.5.18) is uniquely maximized at $\alpha = 0$, it holds that $\mathbb{E}[Z_N(\beta)^2] = \kappa(1+o(1))\mathbb{E}[Z_N(\beta)]^2$ for a constant $\kappa > 1$ (which again comes from the correction to Laplace's method), so at best Paley-Zygmund only yields

$$\liminf_{N\to\infty} \mathbb{P}\left(Z_N \geq \exp\left(N\frac{\beta^2}{2}z(1) - o(N)\right)\right) \geq \underbrace{\kappa^{-1}}_{<1}. \tag{3.5.24}$$

Luckily, the weak lower bound (3.5.21) can be strengthened to our goal (3.5.2) without using the precise asymptotic (3.5.22). Instead, general concentration results for Lipschitz functions of independent Gaussians can be used to derive (3.5.2) from (3.5.21).

The rest of this section deals with the technicalities that are needed to finish the proof of Proposition 3.5.1 (deriving (3.5.2) from (3.5.21)). Unless you are particular interested in the details, feel free to skip ahead to Section 3.6, where the condition (3.5.1) is investigated.

3.5.1 Remaining technicality: Strengthening (3.5.21) to (3.5.2)

To strengthen the weak lower bound (3.5.21) to the desired (3.5.2) we use the following standard general concentration result.

Theorem 3.5.7 (Concentration of Lipschitz functions of independent Gaussians). *Let $X_1, X_2, \ldots$ be i.i.d. standard Gaussian random variables. Let $d \geq 1$ and let $F : \mathbb{R}^d \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant at most $L > 0$. Then*

$$\mathbb{P}\left(|F(X_1,\ldots,X_d) - \mathbb{E}[F(X_1,\ldots,X_d)]| \geq u\right) \leq 2\exp\left(-\frac{u^2}{2L}\right) \quad \textit{for all} \quad u \geq 0. \tag{3.5.25}$$

Theorem 3.5.7 is useful to us because it turns out that the free energy of a Hamiltonian constructed from couplings $J_{i_1,\dots,i_p}$ is Lipschitz in these couplings. To formally state this, define the map

$$\tilde{H}_N^p : \mathbb{R}^{N^p} \times S_{N-1} \to \mathbb{R}, \quad \tilde{H}_N^p(\mathbf{J}, \sigma) = \sum_{i_1,\dots,i_p=1}^{N} N^{-\frac{p-1}{2}} \mathbf{J}_{i_1,\dots,i_p} \sigma_{i_1} \dots \sigma_{i_p}. \tag{3.5.26}$$

For any $z(x) = \sum_{p=0}^{P} a_p x^p$ with $P < \infty$ define

$$\tilde{H}_N : \mathbb{R}^{1+N+N^2+\dots+N^P} \to \mathbb{R}, \quad \tilde{H}_N((\mathbf{J}^0, \dots, \mathbf{J}^P), \sigma) = \sum_{p=0}^{P} \sqrt{a_p} \tilde{H}_N^p(\mathbf{J}^p, \sigma). \tag{3.5.27}$$

Also define

$$\tilde{F}_N : \mathbb{R}^{1+N+N^2+\dots+N^P} \to \mathbb{R} \quad \text{by} \quad \tilde{F}_N(\mathbf{J}) = \frac{1}{N} \log Q_N \left[\exp(\beta \tilde{H}_N(\mathbf{J}, \sigma)) \right]. \tag{3.5.28}$$

Note that the free energy $F_N(\beta)$ of the mixed p-spin Hamiltonian with covariance function $z(x)$ satisfies

$$\begin{gathered} F_N(\beta) \overset{\text{law}}{=} \tilde{F}_N(\mathbf{J}) \\ \text{provided } J^p_{i_1,\dots,i_p} \text{ are i.i.d. standard Gaussians.} \end{gathered} \tag{3.5.29}$$

Lemma 3.5.8. *For all $P < \infty, z(x), \beta \geq 0, N \geq 1$ and any probability measure Q_N on S_{N-1}, the function $\tilde{F}_N$ is Lipschitz with Lipschitz constant at most $\beta\sqrt{z(1)} N^{-\frac{1}{2}}$.*

Proof. It suffices to show that

$$\left| \nabla_{\mathbf{J}} \tilde{F}_N(\mathbf{J}) \right| \leq \frac{\beta \sqrt{z(1)}}{\sqrt{N}} \quad \text{for all} \quad \mathbf{J} \in \mathbb{R}^{N^p}, \tag{3.5.30}$$

where $\nabla_{\mathbf{J}} \tilde{F}_N(\mathbf{J}) \in \mathbb{R}^{N^p}$ denotes the gradient of $\tilde{F}_N$ w.r.t to $\mathbf{J} \in \mathbb{R}^{1+N+N^2+\dots+N^P}$.

From (3.5.27)-(3.5.28) it follows that

$$\begin{aligned} \partial_{\mathbf{J}^p_{i_1\dots i_p}} \tilde{F}_N(\mathbf{J}) &= \tfrac{\sqrt{a_p}}{N} Q_N \left[\beta \left(\partial_{\mathbf{J}^p_{i_1\dots i_p}} \tilde{H}_N(\mathbf{J}, \sigma) \right) \exp(\beta \tilde{H}_N(\mathbf{J}, \sigma)) \right] \left(Q_N \left[\exp(\beta \tilde{H}_N(\mathbf{J}, \sigma)) \right] \right)^{-1} \\ &= \tfrac{\beta\sqrt{a_p}}{N} \tilde{G}_N \left(\partial_{\mathbf{J}^p_{i_1\dots i_p}} \tilde{H}_N(\mathbf{J}, \sigma); \mathbf{J} \right), \end{aligned} \tag{3.5.31}$$

where $\tilde{G}_N(A; \mathbf{J}) \propto Q_N \left[1_A \exp(\beta \tilde{H}_N(\mathbf{J}, \sigma)) \right]$ denotes the Gibbs measure of the Hamiltonian $\sigma \to \tilde{H}_N(\mathbf{J}, \sigma)$. The function $\tilde{H}_N(\mathbf{J}, \sigma)$ is linear in $\mathbf{J}$, and trivially

$$\partial_{\mathbf{J}^p_{i_1\dots i_p}} \tilde{H}_N(\mathbf{J}, \sigma) = N^{-\frac{p+1}{2}} \sigma_{i_1} \dots \sigma_{i_p}, \quad \text{for all} \quad p \in \{0, \dots, P\}, i_1, \dots, i_p \in \{1, \dots, N\}. \tag{3.5.32}$$

Thus

$$\partial_{\mathbf{J}^p_{i_1\dots i_p}} \tilde{F}_N(\mathbf{J}) = \beta \sqrt{a_p} N^{-\frac{p}{2}} \tilde{G}_N \left(\sigma_{i_1} \dots \sigma_{i_p}; \mathbf{J} \right). \tag{3.5.33}$$

By Jensen's inequality $\tilde{G}_N(\sigma_{i_1}\dots\sigma_{i_p};\mathbf{J})^2 \le \tilde{G}_N(\sigma_{i_1}^2\dots\sigma_{i_p}^2;\mathbf{J})^2$ for all $\sigma \in S_{N-1}$ (for $\sigma \in \{-1,1\}^N$ both sides actually trivially equal one). Thus

$$\left(\partial_{\mathbf{J}^p_{i_1\dots i_p}} \tilde{F}_N(\mathbf{J})\right)^2 \le \beta^2 a_p N^{-p} \tilde{G}_N(\sigma_{i_1}^2\dots\sigma_{i_p}^2;\mathbf{J}). \tag{3.5.34}$$

Summing (3.5.34) over $i_1,\dots,i_p$ and then p we obtain

$$\left|\nabla_{\mathbf{J}} \tilde{F}_N(\mathbf{J})\right|^2 \le \beta^2 \sum_{p=0}^{P} a_p N^{-p} \tilde{G}_N\left(\sum_{i_1,\dots,i_p=1}^{N} \sigma_{i_1}^2\dots\sigma_{i_p}^2;\mathbf{J}\right). \tag{3.5.35}$$

For $\sigma \in S_{N-1}$

$$\sum_{i_1,\dots,i_p=1}^{N} \sigma_{i_1}^2\dots\sigma_{i_p}^2 = |\sigma|^{2p} = N^p, \tag{3.5.36}$$

so

$$\left|\nabla_{\mathbf{J}} \tilde{F}_N(\mathbf{J})\right|^2 \le \beta^2 \sum_{p=0}^{P} a_p N^{-(p+1)} N^p = \frac{\beta^2}{N} \sum_{p=0}^{P} a_p = \frac{\beta^2}{N} z(1). \tag{3.5.37}$$

This proves (3.5.30), and therefore that $\tilde{F}_N$ is $\beta\sqrt{z(1)}N^{-1/2}$-Lipschitz. □

Corollary 3.5.9. *For all $P < \infty, z(x) = \sum_{p=0}^{P} a_p x^p, \beta \ge 0, N \ge 1$ and any probability measure Q_N on S_{N-1}*

$$\mathbb{P}\left(|F_N(\beta) - \mathbb{E}(F_N(\beta))| \ge u\right) \le \exp\left(-N\frac{u^2}{2\beta^2 z(1)}\right) \quad \textit{for all} \quad N \ge 1, \beta \ge 0, u \ge 0. \tag{3.5.38}$$

Proof. This follows from (3.5.29), Lemma 3.5.8 and Theorem 3.5.7. □

Lemma 3.5.10. *If for some $\beta \ge 0$ the estimate (3.5.21) holds then*

$$\liminf_{N\to\infty} \mathbb{E}[F_N(\beta)] \ge \frac{\beta^2}{2} z(1). \tag{3.5.39}$$

Remark 3.5.11. We already know from Corollary 3.4.2 that $\mathbb{E}[F_N(\beta)] \le \frac{\beta^2}{2} z(1)$, so if (3.5.21) holds then in fact

$$\lim_{N\to\infty} \mathbb{E}[F_N(\beta)] = \frac{\beta^2}{2} z(1). \tag{3.5.40}$$

Proof. Assume for contradiction that (3.5.39) does not hold, i.e. that $\liminf_{N\to\infty} \mathbb{E}[F_N(\beta)] < \frac{\beta^2}{2} z(1)$. Then there is an $\varepsilon > 0$ such that

$$\mathbb{E}[F_N(\beta)] \le \frac{\beta^2}{2} z(1) - \varepsilon \quad \text{for infinitely many} \quad N. \tag{3.5.41}$$

For such N

$$\mathbb{P}\left(F_N(\beta) \geq \frac{\beta^2}{2}z(1) - \frac{\varepsilon}{2}\right) \leq \mathbb{P}\left(F_N(\beta) \geq \mathbb{E}[F_N(\beta)] + \frac{\varepsilon}{2}\right) \overset{(3.5.38)}{\leq} \exp\left(-N\frac{\varepsilon^2}{8\beta^2 z(1)}\right). \quad (3.5.42)$$

But at the same time we have for N large enough that

$$\mathbb{P}\left(F_N(\beta) \geq \frac{\beta^2}{2}z(1) - \frac{\varepsilon}{2}\right) \geq \mathbb{P}\left(F_N(\beta) \geq \frac{\beta^2}{2}z(1) - cN^{-1}\right) \overset{(3.5.21)}{\geq} \exp(-o(N)). \quad (3.5.43)$$

Note that (3.5.42) and (3.5.43) are in contradiction. Thus (3.5.39) must hold. □

Proof of Proposition 3.5.1. The (weak) estimate (3.5.21) holds under the assumptions of Proposition 3.5.1. Thus by Lemma 3.5.10 it holds for any $\varepsilon > 0$ and N large enough that

$$\frac{\beta^2}{2}z(1) \leq \mathbb{E}[F_N(\beta)] + \frac{\varepsilon}{2}, \quad (3.5.44)$$

implying

$$\mathbb{P}\left(F_N(\beta) \leq \frac{\beta^2}{2}z(1) - \varepsilon\right) \leq \mathbb{P}\left(F_N(\beta) \leq \mathbb{E}[F_N(\beta)] - \frac{\varepsilon}{2}\right). \quad (3.5.45)$$

By Corollary 3.5.9

$$\mathbb{P}\left(F_N(\beta) \leq \mathbb{E}[F_N(\beta)] - \frac{\varepsilon}{2}\right) \leq 2\exp\left(-N\frac{\varepsilon^2}{8\beta^2 z(1)}\right) \to 1, \quad (3.5.46)$$

so (3.5.2) follows. □

3.6 Analysis of the the second moment condition

Let us now investigate the second moment condition

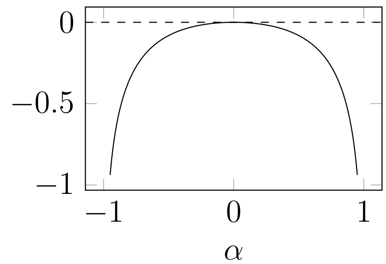
$$\sup_{\alpha \in (-1,1)} \left\{ \beta^2 z(\alpha) - I(\alpha) \right\} \leq 0, \tag{3.6.1}$$

which which we saw in the previous section is sufficient for quenched free energy to be given by the annealed free energy, i.e. for

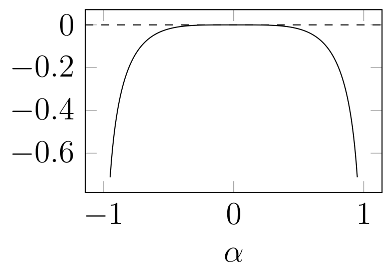
$$F_N(\beta) \xrightarrow{\mathbb{P}} \frac{\beta^2}{2} z(1). \tag{3.6.2}$$

The following figures plot the function $\beta^2 z(\alpha) - I(\alpha)$ for $I = I^{\text{sph}}$some examples of covariance functions $z(\alpha)$.

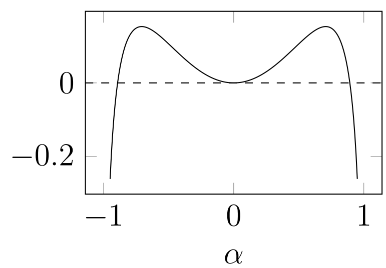
$\beta = \frac{1}{2}$



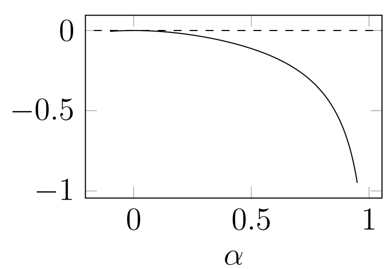
$z(x) = x^2, \beta = \frac{1}{\sqrt{2}}$



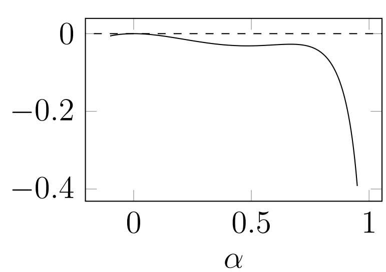
$z(x) = x^2, \beta = 1$



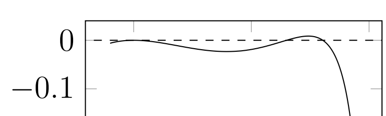
$z(x) = x^2, \beta = \frac{1}{2}$



$z(x) = x^3, \beta = \sqrt{0.9}$



$z(x) = x^3, \beta = 1$



From Figure 3.6.1 we observe that

$$\begin{array}{ccc} & \text{for } z(x) = x^2 \text{ and } I = I^{\pm} \text{ or } I = I^{\rm sph} & \\ \beta \in \left[0, \frac{1}{\sqrt{2}}\right] & \iff & \sup_{\alpha\in(-1,1)} \{\beta^2 z(\alpha) - I(\alpha)\} = 0. \end{array} \tag{3.6.3}$$

The implications in (3.6.3) are easy to prove prove rigorously, see below. From (3.6.3) it follows that

$$F_N(\beta) \xrightarrow{\mathbb{P}} \frac{\beta^2}{2} z(1) \quad \text{for} \quad z(x) = x^2, Q_N \in \{Q_N^{\rm sph}, Q_N^{\pm}\}, \beta \in [0, \frac{1}{\sqrt{2}}). \tag{3.6.4}$$

It turns out that $\beta = \frac{1}{\sqrt{2}}$ is in fact the critical inverse temperature β_c for the models in (3.6.4); that is, for $\beta > \frac{1}{\sqrt{2}}$ it holds that $\limsup_{N\to\infty} F_N(\beta) < \frac{\beta^2}{2} z(1)$. We will not cover the proof of this fact. The situation in (3.6.4) where the sec mom condition is equivalent to the "sharp" high-temperature condition $\beta \leq \beta_c$ is unusual. Define for $z(x) = \sum_{p=2}^P a_p x^2$ and $I \in \{I^{\pm}, I^{\rm sph}\}$

$$\beta_{\rm 2nd} = \beta_{\rm 2nd}(z, I) := \sup\{\beta \geq 0 : \beta^2 z(\alpha) - I(\alpha) \leq 0 \, \forall \alpha \in (-1, 1)\}, \tag{3.6.5}$$

so that

$$\beta \in [0, \beta_{\rm 2nd}(z, I)] \quad \iff \quad \sup_{\alpha\in(-1,1)} \left\{\beta^2 z(\alpha) - I(\alpha)\right\} = 0. \tag{3.6.6}$$

For the 2-spin models in (3.6.4) we have $\beta_{\rm 2nd}(z, I) = \beta_c(z, I)$, but the much more common situation is that $\beta_{\rm 2nd}(z, I) < \beta_c(z, I)$.

The following general lemmas are easily proved by elementary computations.

Lemma 3.6.1. *Let $P \geq 2$ and let $z(x) = \sum_{p=2}^P a_p x^p$ be a covariance function with $a_0 = a_1 = 0$ and $z(1) > 0$. For each $N \geq 1$, let H_N be a mixed p-spin Hamiltonian with covariance function z and without external field ($h = 0$). Let $I = I^{\pm}$ (Ising model) or $I = I^{\rm sph}$ (spherical model). Then For all $z(x) = \sum_{p\geq 2} a_p x^p s.t.\ z(1) < \infty$ and $I \in \{I^{\pm}, I^{\rm sph}\}$ it holds that $\beta_{\rm 2nd}(z, I) > 0$. In fact*

$$\beta_{\rm 2nd}(z, I) = \inf_{\alpha\in(-1,1)\setminus\{0\}} \sqrt{\frac{I(\alpha)}{z(\alpha)}} > 0. \tag{3.6.7}$$

Lemma 3.6.2. *If $z(x) = a_2 x^2$ for $a_2 > 0$ then holds that*

$$\beta_{\rm 2nd}(z, I) = \frac{1}{\sqrt{2a_2}}. \tag{3.6.8}$$

More generally, (3.6.1) holds if $Q_N = Q_N^{\rm sph}$ and $z(x) = a_2 x^2 + \sum_{p\in\{4,6,\dots\}} a_p x^p$ for $0 \leq a_p \leq \frac{a_2}{p}$, or if $Q_N = Q_N^{\pm}$ and $z(x) = x^2 + \sum_{p\in\{4,6,\dots\}} a_p x^p$ for $0 \leq a_p \leq \frac{a_2}{p(p-1)}$.

3.7 Geometric derivation of TAP free energy

TAP free energy of Ising mixed p-spin model

$$F^{\pm}_{\mathrm{TAP}}(m) = \beta H_N(m) - \sum_{i=1}^{N} I^{\pm}(m_i) + N\frac{\beta^2}{2}\left(z(1) - z'(q_m)(1-q_m) - z(q_m)\right), \tag{3.7.1}$$

where

$$q_m = \frac{|m|^2}{N}. \tag{3.7.2}$$

TAP free energy of spherical mixed p-spin model

$$F^{\pm}_{\mathrm{TAP}}(m) = \beta H_N(m) - I^{\mathrm{sph}}(1-q_m) + N\frac{\beta^2}{2}\left(z(1) - z'(q_m)(1-q_m) - z(q_m)\right). \tag{3.7.3}$$

TBC

3.8 Sketch of TAP upper bound for free energy

TBC

Chapter 4

Appendix

Proof of Lemma 2.2.1. We first show the upper bound

$$Q_N\left[\left\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \geq \alpha\right\}\right] \leq \exp(-NI_\pm(\alpha)) \quad \text{for} \quad \alpha \in [0,1], N \geq 1. \tag{4.0.1}$$

We have $\sigma \cdot u = \sum_{i=1}^N \sigma_i$, s

$$Q_N\left[\left\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \geq \alpha\right\}\right] = Q_N\left[\sum_{i=1}^N \sigma_i \geq N\alpha\right]. \tag{4.0.2}$$

Under Q_N the $\sigma_1, \dots, \sigma_N$ are i.i.d. such that $Q_N(\sigma_i = 1) = Q_N(\sigma_i = -1) = \frac{1}{2}$. Thus

$$Q_N[\exp(\lambda\sigma_i)] = \cosh(\lambda) \quad \text{for all} \quad \lambda \in \mathbb{R}. \tag{4.0.3}$$

By the exponential Chebyshev inequality we obtain

$$Q_N\left[\sum_{i=1}^N \sigma_i \geq N\alpha\right] \leq \exp\left(N \inf_{\lambda \geq 0} g(\alpha, \lambda)\right). \tag{4.0.4}$$

where

$$g(\alpha, \lambda) = \lambda\alpha - \log\cosh(\alpha), \quad \alpha \in [0,1], \lambda \in [0, \infty). \tag{4.0.5}$$

For every $\alpha \in [0,1)$, the function $\lambda \to g(\alpha, \lambda)$ is uniquely minimized by $\lambda(\alpha) := \operatorname{atanh}(\alpha)$, and

$$g(\alpha, \lambda(\alpha)) = I(\alpha) \quad \text{for all} \quad \alpha \in (-1, 1). \tag{4.0.6}$$

This proves (4.0.1) for $\alpha \in [0,1)$. For $\alpha = 1$ equality holds, since $Q_N\left[\sum_{i=1}^N \sigma_i \geq N\right] = Q_N\left[\sum_{i=1}^N \sigma_i = N\right] = 2^{-N}$ and $\exp(-NI_\pm(1)) = \exp(-N\log 2) = 2^{-N}$.

For the lower bound, let $X_i = \frac{\sigma_i + 1}{2}$ be i.i.d. Bernoulli r.v.s. with parameter $p = \frac{1}{2}$. Let $p(\alpha) = N^{-1}\lceil\frac{N(1+\alpha)}{2}\rceil$ denote $\frac{1+\alpha}{2}$ rounded up to the closest multiple of N^{-1}, and note that

$$Q_N\left[\sum_{i=1}^N \sigma_i \geq N\alpha\right] = Q_N\left[\sum_{i=1}^N X_i \geq \frac{N(1+\alpha)}{2}\right] \geq Q_N\left[\sum_{i=1}^N X_i = Np(\alpha)\right] = \frac{1}{2^N}\binom{N}{Np(\alpha)}, \tag{4.0.7}$$

where $\lceil x \rceil$ denote x rounded up to the nearest integer. Let

$$I(p) = p \log p + (1-p)\log(1-p), \tag{4.0.8}$$

so that

$$I(p) = I_{\pm}(2p-1) - \log 2 \quad \text{for} \quad p \in [0,1]. \tag{4.0.9}$$

Using Stirling's formula

$$\log(n!) = n \log n - n + O(\log n), \tag{4.0.10}$$

we obtain

$$\log \binom{N}{Np(\alpha)} = -NI(p(\alpha)) + O(\log N). \tag{4.0.11}$$

Note that

$$\left| I\left(\frac{k}{N}\right) - I\left(\frac{k+1}{N}\right) \right| \leq \left| I(1) - I\left(\frac{N-1}{N}\right) \right| = \log N.. \tag{4.0.12}$$

$$|I_{\pm}(2p(\alpha)-1) - I_{\pm}(\alpha)| \leq \frac{\log N}{N}, \tag{4.0.13}$$

so

$$\log \binom{N}{Np(\alpha)} = N\left\{I_{\pm}(\alpha) + \log 2\right\} + O(\log N). \tag{4.0.14}$$

$$|(2p(\alpha)-1) - \alpha| \leq N^{-1}, \tag{4.0.15}$$

and $\varepsilon > 0$ the function $I_{\pm}$ is Lipschitz on $[0, 1-\varepsilon]$, so

$$I_{\pm}(2p(\alpha)-1) = I_{\pm}(\alpha) + O(N^{-1}). \tag{4.0.16}$$

$$\leq \exp\left(-NI_{\pm}(2p(\alpha)-1) + o(N)\right) \tag{4.0.17}$$

$$\tag{4.0.18}$$

$$-N\log 2 - N \log 2 + \log N \tag{4.0.19}$$

□

Lemma 4.0.1. *Let X be a standard Gaussian random variable. Then exponential moment of X^2 equals*

$$\mathbb{E}[\exp(\lambda X^2)] = \begin{cases} \frac{1}{\sqrt{1-2\lambda}} & \textit{if} \quad \lambda \in \left(-\infty, \frac{1}{2}\right), \\ \infty & \textit{otherwise}. \end{cases} \tag{4.0.20}$$

Proof. Since $\int_{-\infty}^{\infty} e^{\lambda x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d\lambda = \frac{1}{\sqrt{1-2\lambda}}$. □

The next lemma derives large deviation bounds for sums $\sum_{i=1}^{N} X_i^2$ where X_i is standard Gaussian. The large deviation rate function is

$$I(u) = \frac{u - \log u - 1}{2}, \quad u > 0. \tag{4.0.21}$$

Lemma 4.0.2. *Let $X_1, X_2, \dots$ be i.i.d. standard Gaussian random variables. For any $\varepsilon > 0$ and large enough N it holds that*

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^2 \le u\right) = \exp(-NI(u) + o(N)) \quad \textit{uniformly in} \quad u \in [\varepsilon, 1]. \tag{4.0.22}$$

Proof. We first prove the upper bound

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^2 \le u\right) \le \exp(-NI(u)) \quad \text{for all} \quad u \in (0,1], N \ge 1. \tag{4.0.23}$$

Let

$$f(\lambda) = \frac{1}{2}\log(1-2\lambda) \quad \text{and} \quad g(u,\lambda) = f(\lambda) - \lambda u, \quad \lambda \in \left(-\infty, \frac{1}{2}\right), u \in (0,1]. \tag{4.0.24}$$

By (4.0.20)

$$\mathbb{E}\left[\exp\left(\lambda \sum_{i=1}^{N} X_i^2\right)\right] = \exp(Nf(\lambda)) \quad \text{for all} \quad \lambda \in \left(-\infty, \frac{1}{2}\right), \tag{4.0.25}$$

and so by the exponential Chebyshev inequality

$$\mathbb{P}\left(\frac{1}{N}\sum_{i=1}^{N} X_i^2 \le u\right) \le \exp\left(N \inf_{\lambda \le 0} g(u,\lambda)\right) \quad \text{for all} \quad u \in (0,1], \tag{4.0.26}$$

For $u \in (0,1]$ the function $\lambda \to g(u,\lambda)$ is uniquely minimized at

$$\lambda = \lambda(u) := -\frac{1-u}{2u} \quad \text{for which} \quad g(u,\lambda(u)) = -I(u). \tag{4.0.27}$$

This proves (4.0.23).

To prove the lower bound (4.0.22), fix $\varepsilon > 0$ and let $\mathbb{Q}_{N,\lambda}$ be the probability measure such that

$$\mathbb{Q}_{N,\lambda}(A) = \frac{\mathbb{E}[1_A \exp(\lambda \sum_{i=1}^N X_i^2)]}{\mathbb{E}[\exp(\lambda \sum_{i=1}^N X_i^2)]}, \quad \lambda < \frac{1}{2}. \tag{4.0.28}$$

Under the measure $\mathbb{Q}_{N,\lambda}$ the $X_i^2, i = 1, \dots, N$, are i.i.d. with mean $f'(\lambda) = (1-2\lambda)^{-1}$ and variance $f''(\lambda) = 2(1-2\lambda)^{-2}$. If $\lambda = \lambda(w)$ for $w \in (0,1]$, then the mean is $f'(\lambda(w)) = w$ and the variance is $f''(\lambda(w)) = 2w^2$. For N large enough so that $\varepsilon \ge 4N^{-\frac{1}{2}}$ and any $u \in [\varepsilon, 1]$, set $\delta = 2N^{-\frac{1}{2}}$ and define $w(u) = u - \delta$ for $u \in [\varepsilon, 1]$. Note that $w(u) \in [\frac{\varepsilon}{2}, 1)$ for all $u \in [\varepsilon, 1]$. Set $\lambda = \lambda(w(u))$. Then

$$\lambda(w(u)) < \lambda(1) = 0 \quad \text{for all} \quad u \in [\varepsilon, 1]. \tag{4.0.29}$$

Thus with $\lambda = \lambda(w(u))$

$$\begin{aligned} & \mathbb{P}\left(u - 2\delta \le \tfrac{1}{N}\textstyle\sum_{i=1}^N X_i^2 \le u\right) \\ = \quad & \mathbb{Q}_{N,\lambda}\left[1_{\left\{u-2\delta \le \frac{1}{N}\sum_{i=1}^N X_i^2 \le u\right\}} \exp(-\lambda \textstyle\sum_{i=1}^N X_i^2)\right] \exp(Nf(\lambda)) \\ \overset{\lambda(w(u))<0}{\ge} \quad & \mathbb{Q}_{N,\lambda}\left[u - 2\delta \le \tfrac{1}{N}\textstyle\sum_{i=1}^N X_i^2 \le u\right] \exp\left(N\left\{-\lambda(u-2\delta) + f(\lambda)\right\}\right). \end{aligned} \tag{4.0.30}$$

We have

$$\mathbb{Q}_{N,\lambda}\left[u - 2\delta \le \frac{1}{N}\sum_{i=1}^N X_i^2 \le u\right] = \mathbb{Q}_{N,\lambda}\left[-\delta \le \frac{1}{N}\sum_{i=1}^N (X_i^2 - w(u)) \le \delta\right], \tag{4.0.31}$$

where the $X_i^2 - w(u)$ are i.i.d. with mean zero and variance $2w(u)^2$ under $\mathbb{Q}_{N,\lambda}$. By the Chebyshev inequality

$$\mathbb{Q}_{N,\lambda}\left[-\delta \le \frac{1}{N}\sum_{i=1}^N (X_i^2 - w) \le \delta\right] \ge 1 - \frac{2w(u)^2}{N\delta^2} = 1 - \frac{w(u)^2}{2} \ge \frac{1}{2} \quad \text{for all} \quad u \in [\varepsilon, 1]. \tag{4.0.32}$$

Thus from (4.0.30) it follows that

$$\mathbb{P}\left(u - 2\delta \le \frac{1}{N}\sum_{i=1}^N X_i^2 \le u\right) \ge \frac{1}{2}\exp\left(N\left\{-\lambda(u-2\delta) + f(\lambda)\right\}\right) \quad \text{for all} \quad u \in [\varepsilon, 1]. \tag{4.0.33}$$

Furthermore

$$-\lambda(u - 2\delta) + f(\lambda) = g(u,\lambda) + 2\delta\lambda = g(u, \lambda(u-\delta)) + 2\delta\lambda(u-\delta). \tag{4.0.34}$$

For $u \in [\varepsilon, 1]$ it holds that $[u-\delta, u] \in [\frac{\varepsilon}{2}, 1]$. The function $v \to \lambda(v)$ is bounded and Lipschitz in $[\frac{\varepsilon}{2}, 1]$, and $\lambda([\frac{\varepsilon}{2}, 1]) \subset [\lambda(\frac{\varepsilon}{2}), 0]$. Furthermore g is Lipschitz on $[0,1] \times [\lambda(\frac{\varepsilon}{2}), 0]$. Thus the r.h.s. of (4.0.34) equals

$$g(u, \lambda(u)) + 2\delta\lambda(u) + O(\delta) = -I(u) + O(\delta) \quad \text{uniformly for} \quad u \in [\varepsilon, 1]. \tag{4.0.35}$$

Combining these with (4.0.33), the lower bound of (4.0.22) follows. □

Proof of Lemma 2.2.2. For $\alpha = 0$ the claim is obvious. Also by the rotational invariance of Q_N we can w.l.o.g. set $u = (0, \dots, 0, \sqrt{N}) \in \mathbb{R}^N$.

We first prove the upper bound

$$Q_N\left[\left\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \ge \alpha\right\}\right] \le \exp\left(\frac{N-1}{2}\log(1-\alpha^2) + \frac{1}{2}\log(eN)\right) \quad \text{for all} \quad \alpha \in [0,1), N \ge 1. \tag{4.0.36}$$

Let $X_1, X_2, \ldots$ be i.i.d standard Gaussian random variables. Let $V_N = (X_1, \ldots, X_N), N \geq 1$ be Gaussian vectors with standard independent components. By the rotational symmetry of the law of V_N, the r.v. $\frac{\sqrt{N}}{|V_N|}V_N$ is uniform on S_{N-1}. Thus

$$Q_N\left[\left\{\sigma \in S_{N-1} : \frac{\sigma \cdot u}{N} \geq \alpha\right\}\right] = \mathbb{P}\left(\frac{\frac{\sqrt{N}}{|V_N|}V_N \cdot u}{N} \geq \alpha\right) = \mathbb{P}\left(\frac{X_N}{\sqrt{X_1^2 + \ldots + X_N^2}} \geq \alpha\right) \tag{4.0.37}$$

for all $\alpha \in (0,1)$ and $N \geq 1$. The r.h.s. in turn equals

$$\mathbb{P}\left(\frac{1-\alpha^2}{\alpha^2}X_N^2 \geq X_1^2 + \ldots + X_{N-1}^2\right). \tag{4.0.38}$$

By the exponential Chebyshev inequality this is abounded above by

$$\inf_{\lambda \geq 0} \mathbb{E}\left[\exp\left(\lambda\left\{\frac{1-\alpha^2}{\alpha^2}X_N^2 - (X_1^2 + \ldots + X_{N-1}^2)\right\}\right)\right]. \tag{4.0.39}$$

By (4.0.20) this equals

$$\inf_{\lambda \in [0, \frac{1}{2}\frac{\alpha^2}{1-\alpha^2})}\left\{\sqrt{\frac{1}{1-2\lambda\frac{1-\alpha^2}{\alpha^2}}}\exp\left(-\frac{N-1}{2}\log(1+2\lambda)\right)\right\}. \tag{4.0.40}$$

For $\lambda = (1-N^{-1})\frac{1}{2}\frac{\alpha^2}{1-\alpha^2}$ we have

$$1 + 2\lambda = 1 + (1-N^{-1})\frac{\alpha^2}{1-\alpha^2} = \frac{1-N^{-1}\alpha^2}{1-\alpha^2}, \tag{4.0.41}$$

and

$$1 - 2\lambda\frac{1-\alpha^2}{\alpha^2} = 1 - (1-N^{-1}) = N^{-1}. \tag{4.0.42}$$

Using this λ we obtain that (4.0.38) is at most

$$\exp\left(\frac{N-1}{2}\log(1-\alpha^2) - \frac{N-1}{2}\log(1-N^{-1}\alpha^2)\right), \tag{4.0.43}$$

for all $\varepsilon \in (0,1)$. Also

$$-\frac{N-1}{2}\log(1-N^{-1}\alpha^2) \leq -\frac{N-1}{2}\log(1-N^{-1}) \leq \frac{N-1}{2}N^{-1} \leq \frac{1}{2}. \tag{4.0.44}$$

This proves (4.0.36).

In turn, (4.0.36) implies the upper bound of (2.2.5), because for any fixed $\varepsilon \in (0,1)$ we have $\frac{1}{2}\log(1-\alpha^2) = o(N)$ uniformly in $\alpha \in [0, 1-\varepsilon]$. To prove the lower bound of (2.2.5), set

$$u = u(\alpha) := 1 - \alpha^2 \tag{4.0.45}$$

and note that

$$\begin{aligned}
&\mathbb{P}\left(\tfrac{1-\alpha^2}{\alpha^2}X_N^2 \geq X_1^2 + \ldots + X_{N-1}^2\right) \\
&\geq \mathbb{P}\left(\tfrac{1-\alpha^2}{\alpha^2}X_N^2 \geq uN \geq u(N-1) \geq X_1^2 + \ldots + X_{N-1}^2\right) \\
&= \mathbb{P}\left(\tfrac{1-\alpha^2}{\alpha^2}X_N^2 \geq uN\right)\mathbb{P}\left(u(N-1) \geq X_1^2 + \ldots + X_{N-1}^2\right),
\end{aligned} \tag{4.0.46}$$

By (4.0.22)

$$\mathbb{P}\left(u(N-1) \geq X_1^2 + \ldots + X_{N-1}^2\right) \geq \exp(-NI(u) + o(N)), \tag{4.0.47}$$

uniformly in $\alpha \in [0, 1-\varepsilon]$. Using the Gaussian tail inequality

$$\mathbb{P}(X_N \geq x) \geq \frac{1}{2x\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for} \quad x \geq \sqrt{2}, \tag{4.0.48}$$

we obtain

$$\mathbb{P}\left(\frac{1-\alpha^2}{\alpha^2}X_N^2 \geq uN\right) \geq \exp\left(-N\frac{u}{2}\frac{\alpha^2}{1-\alpha^2} + o(N)\right), \tag{4.0.49}$$

uniformly in $\alpha \in [0, 1-\varepsilon]$. Note that

$$-I(u) - \frac{u}{2}\frac{\alpha^2}{1-\alpha^2} \overset{u=1-\alpha^2}{=} \frac{1}{2}\log(1-\alpha^2) \quad \text{for all} \quad \alpha \in [0, 1). \tag{4.0.50}$$

From this the lower bound of (2.2.5) follows. $\square$