Continuation

Mixed p-spin FE when $h > 0$

Geometric decomposition

TAP Free Energy
(Thouless-Anderson-Palmer)

Recap: Mixed p-spin Hamiltonian

$a_p \geq 0$

Covariance function: $$z(x) = \sum_{p \geq 0} a_p x^p$$

Mixed p-spin Hamiltonian: H_N

- Gaussian proc. on sphere
- $\mathbb{E}[H_N(\sigma)] = 0$
- $\mathbb{E}[H_N(\sigma) H_N(\tau)] = N z\left(\frac{\sigma \cdot \tau}{N}\right)$

Recap: Annealed Free Energy

$$\mathbb{E}\big(Q_N\big[\exp\big(\beta H_N(\sigma)\big)\big]\big) \quad = \quad \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

$$Q_N\big[\exp\big(\beta H_N(\sigma)\big)\big] \quad \widetilde{\leq} \quad \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

Recap: Annealed Free Energy

$$\left.\begin{array}{c} a_0 \neq 0 \\ \textbf{\textit{or}} \\ a_1 \neq 0 \end{array}\right\} \implies Q_N\left[\exp\left(\beta H_N(\sigma)\right)\right] \gtrsim \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

$$\left.\begin{array}{c} a_0 = a_1 = 0 \\ \textbf{\textit{and}} \\ \beta \leq \beta_c(z) \\ \textbf{\textit{and}} \\ Q_N \text{ unif. on} \\ \{-1,1\}^N \text{ or } S_{N-1} \end{array}\right\} \implies Q_N\left[\exp\left(\beta H_N(\sigma)\right)\right] \cong \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

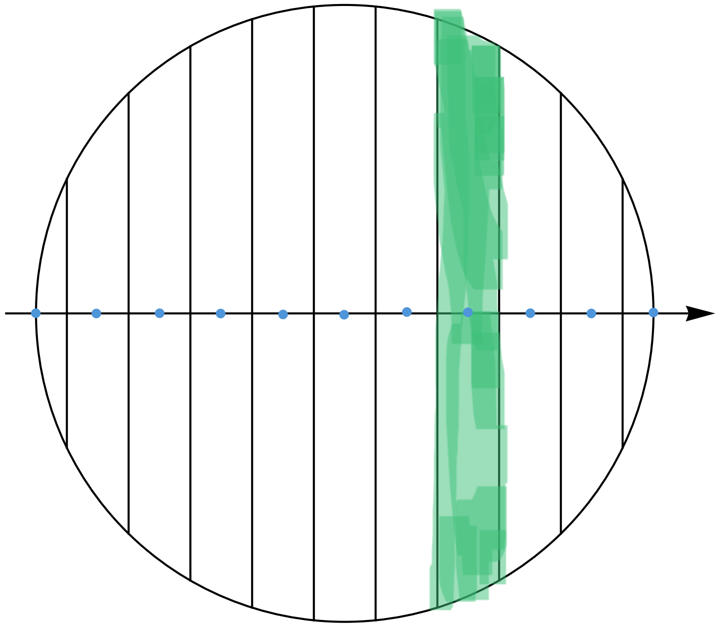
Recap: Geometric decomposition

Configuration space: $S_{N-1} =$ sphere of radius $\sqrt{N}$ in $\mathbb{R}^N$

Reference measure: $Q_N =$ uniform prob. on S_{N-1} .

Hamiltonian: $H_N : S_{N-1} \to \mathbb{R}$

..with ext. field: $H_N^h(\sigma) = H_N(\sigma) + h(\sigma \cdot u)$

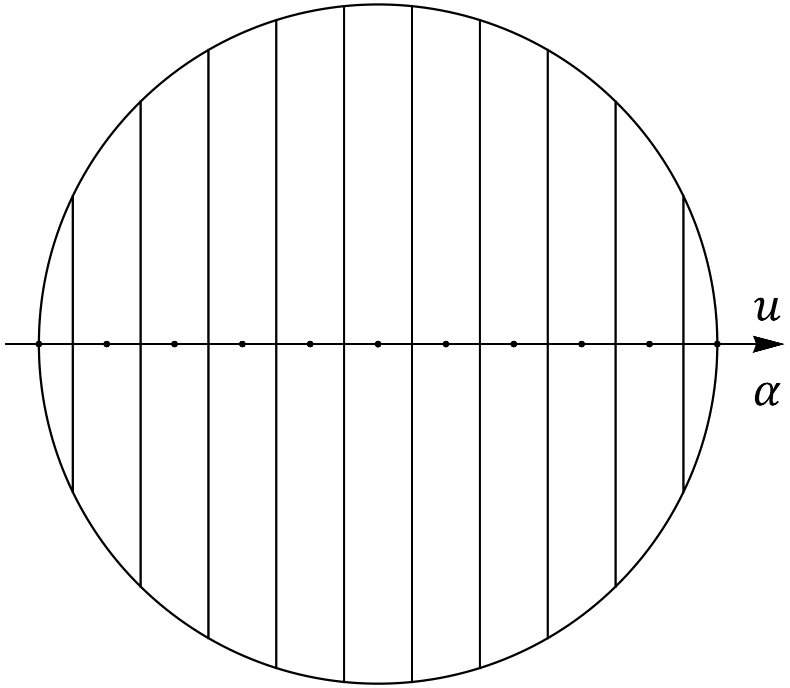


$$D_\alpha := \left\{ \sigma \in S_{N-1} : \left| \frac{\sigma \cdot u}{N} - \alpha \right| \leq N^{-1/3} \right\}$$

$$A := \left(N^{-1/3}\mathbb{Z}\right) \cap (-1,1)$$

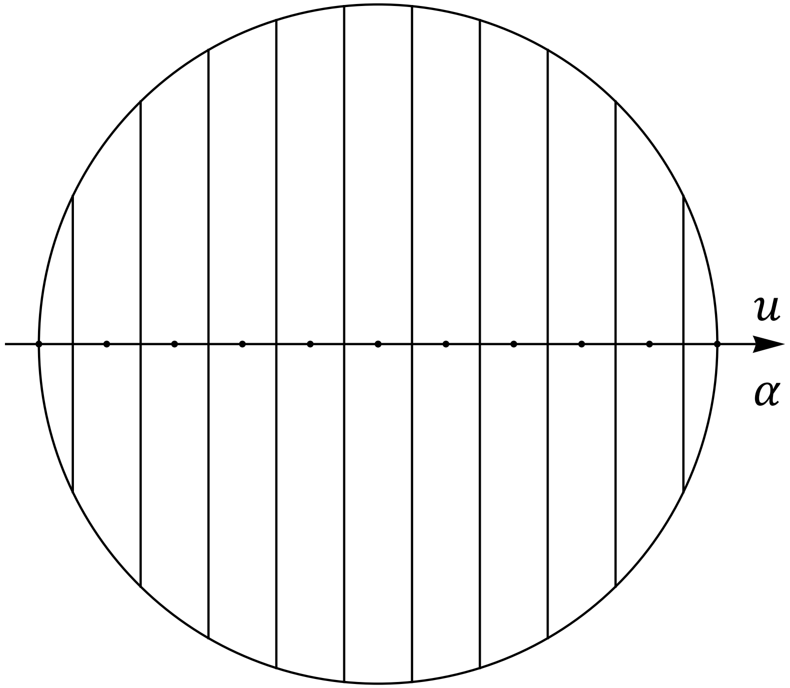
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

$$= \sum_{\alpha \in A} Q_N\left[1_{D_\alpha} \exp\left(\beta H_N^h(\sigma)\right)\right]$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$



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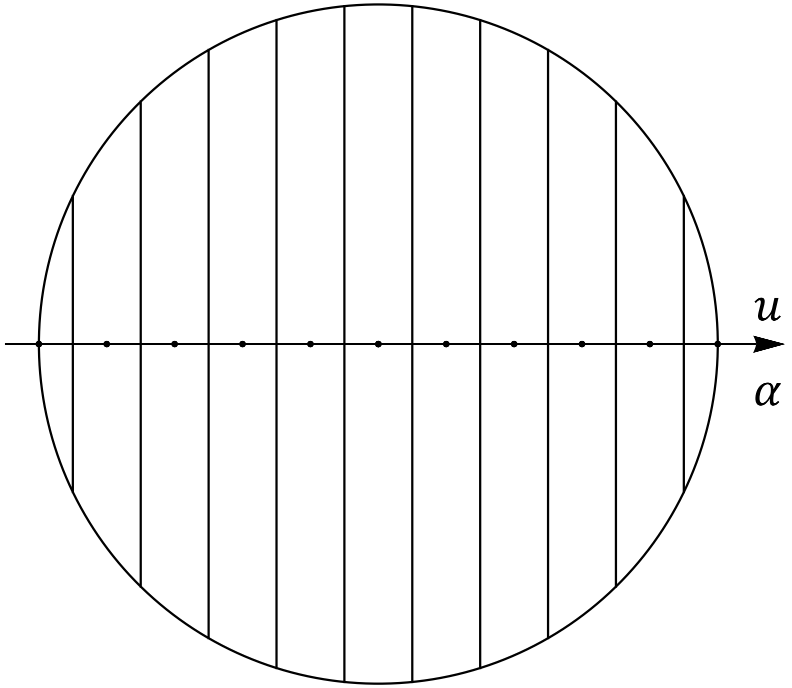
$$\cong$$

$$\exp\left(\frac{N}{2}\log(1-\alpha^2)\right)$$



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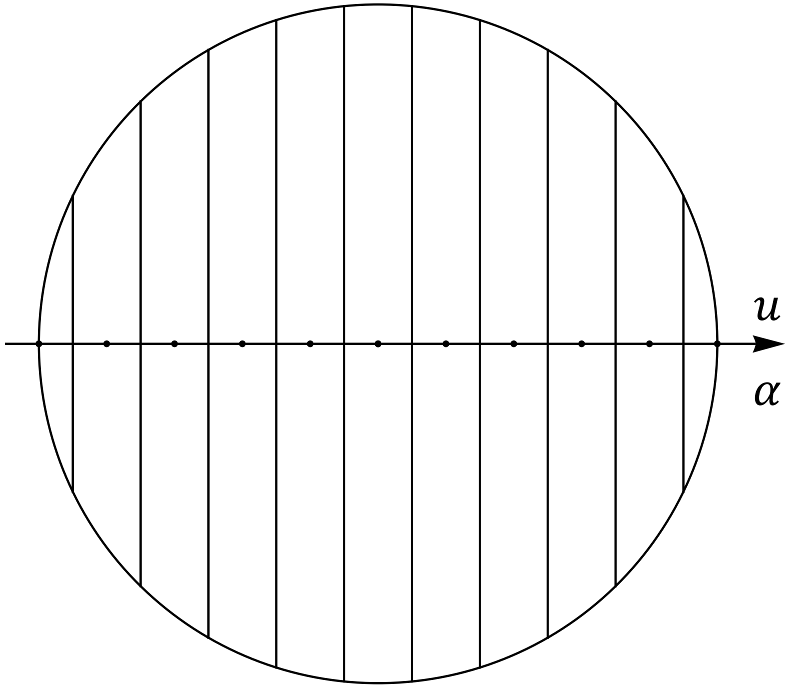


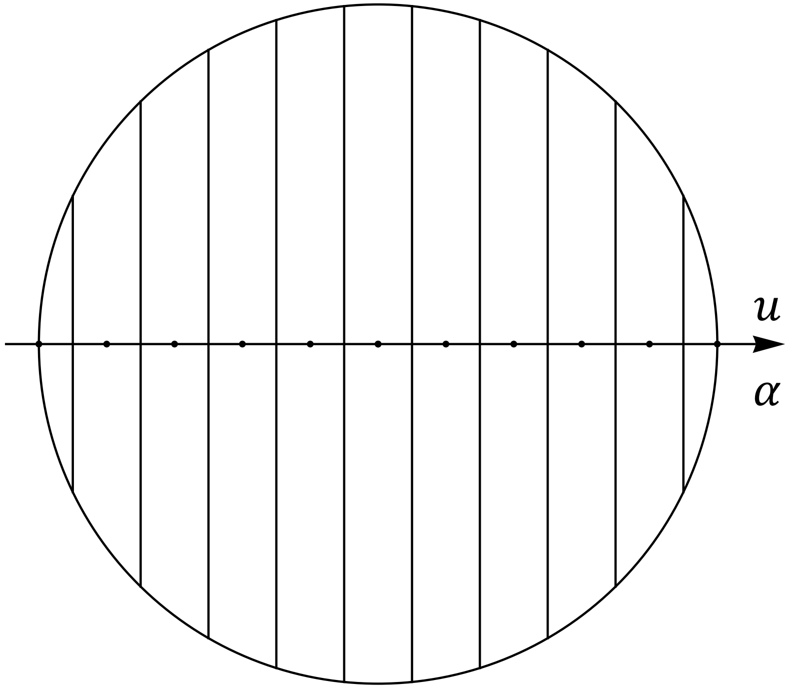
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Curie-Weiss

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha] = \exp(N\beta\alpha^2)$$

$$Q_N[D_\alpha] \cong \exp\left(\frac{N}{2}\log(1-\alpha^2)\right)$$



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Curie-Weiss

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

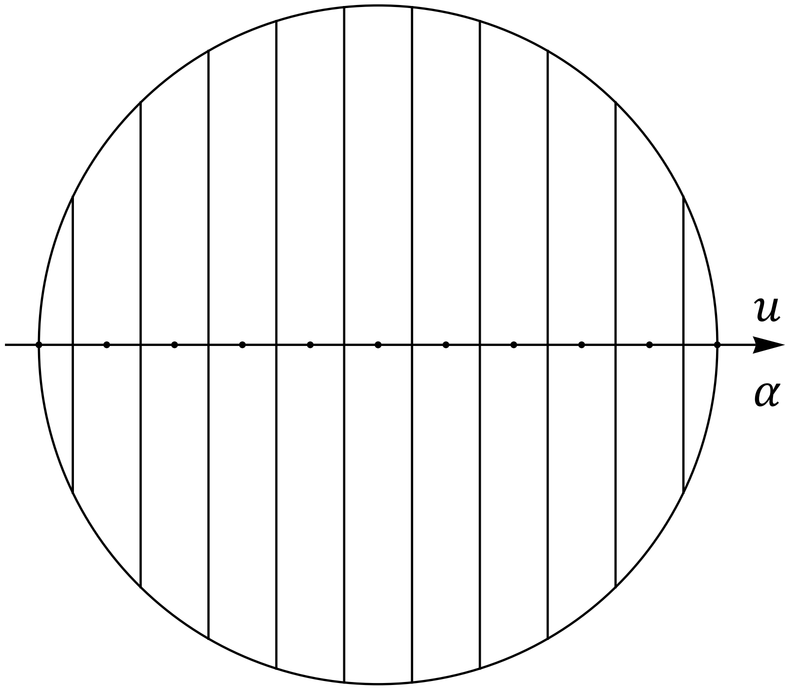
$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] = \exp\left(N\beta\alpha^2\right)$$

$$Q_N[D_\alpha] \cong \exp\left(\frac{N}{2}\log\left(1-\alpha^2\right)\right)$$

$$\exp\left(N\underbrace{\left\{\beta\alpha^2 + \beta h\alpha + \frac{1}{2}\log\left(1-\alpha^2\right)\right\}}_{F(\alpha)}\right)$$

$$Z_N \cong \sum_{\alpha \in A} \exp(NF(\alpha))$$

$$F_N \quad \to \quad \sup_{\alpha \in (-1,1)} \; F(\alpha)$$



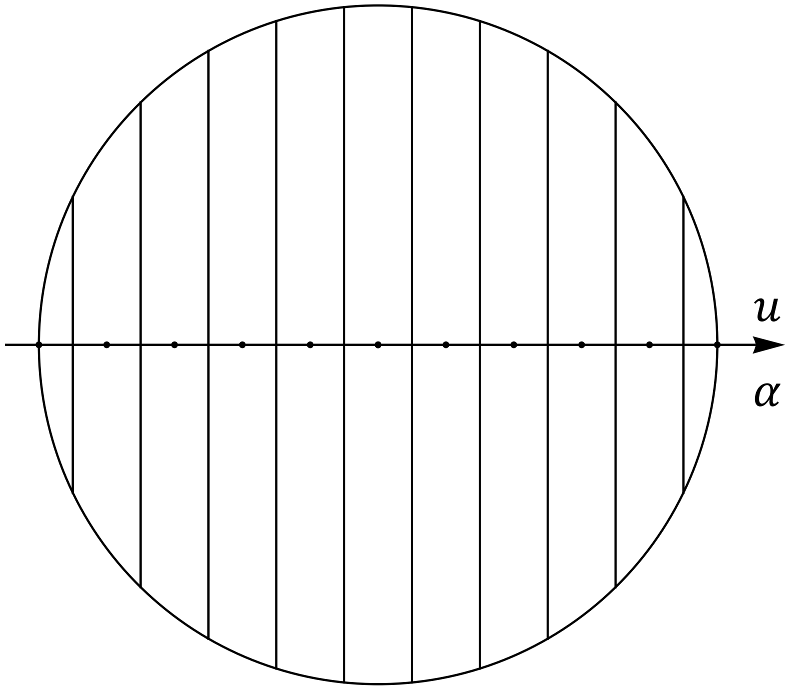
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Mixed p-spin covar $z(x)$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

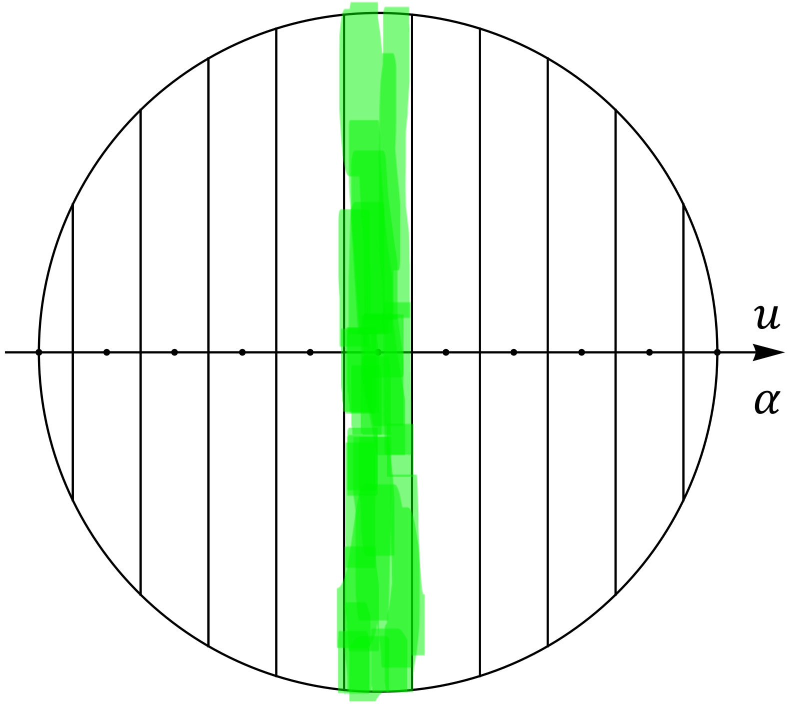
$$\cong$$

$$\exp\left(\frac{N}{2}\log\left(1-\alpha^2\right)\right)$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

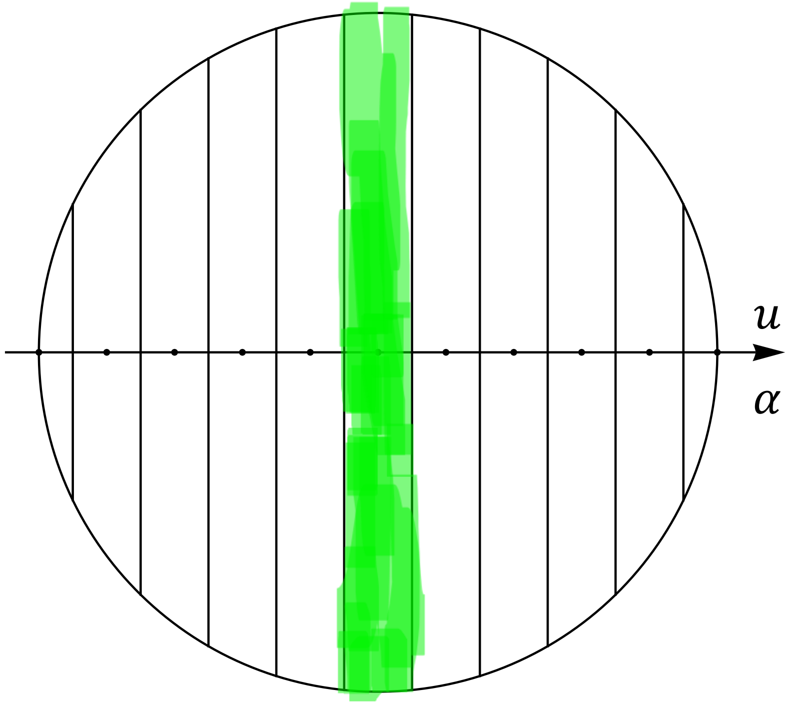
$$Z_N(D_{\alpha}) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_{\alpha}] \times \exp(N\beta h\alpha) \times Q_N[D_{\alpha}]$$



$\alpha = 0$

$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

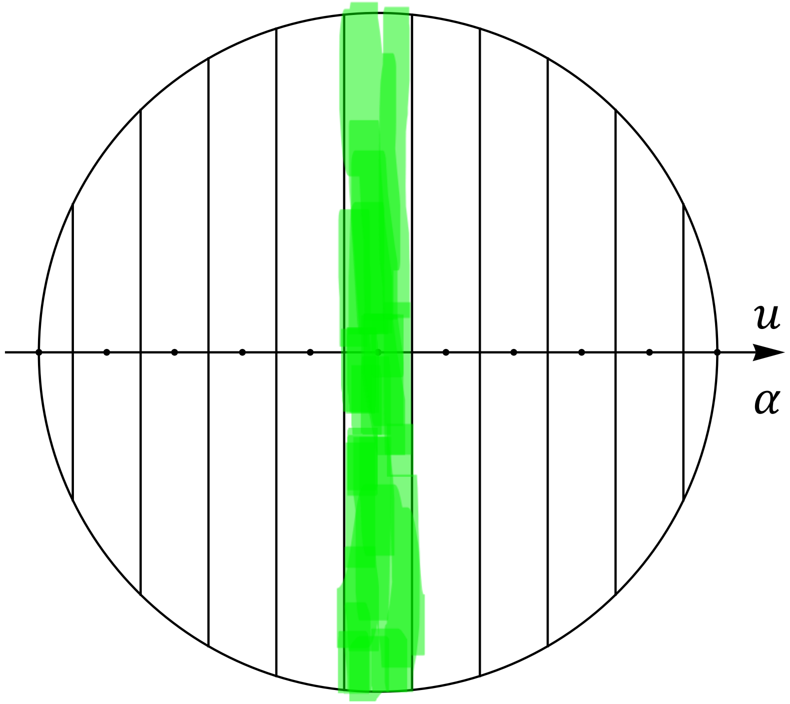
$$Z_N(D_0) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_0] \times \exp(N\beta h \cdot 0) \times Q_N[D_0]$$



$\alpha = 0$

$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

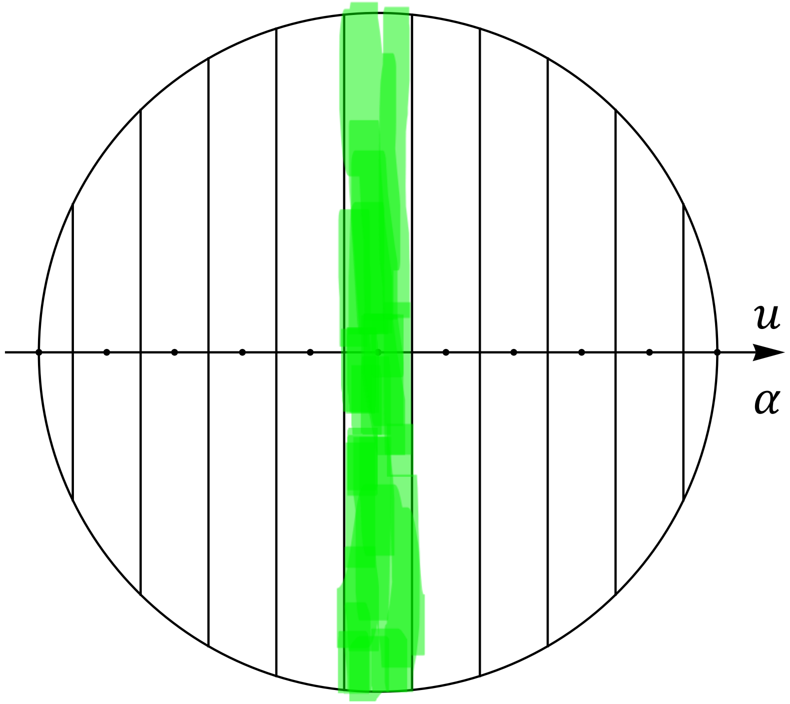
$$Z_N(D_0) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_0] \times \underbrace{\exp(N\beta h \cdot 0)}_{=1} \times \underbrace{Q_N[D_0]}_{\cong\ 1}$$



$\alpha = 0$

$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

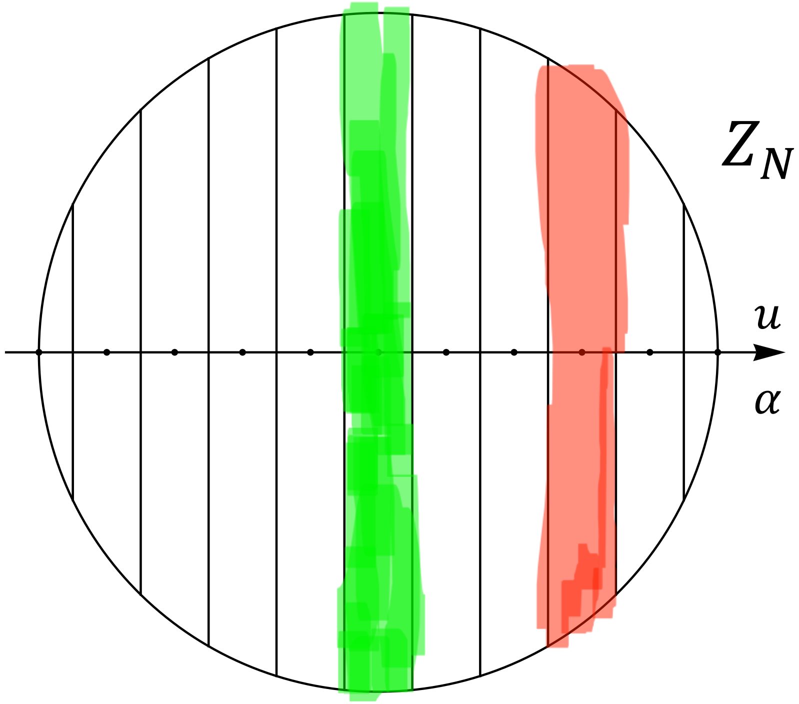
$$Z_N(D_0) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_0] \lesssim \exp\left(N\frac{\beta^2}{2}z(1)\right)$$



$\alpha = 0$

$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

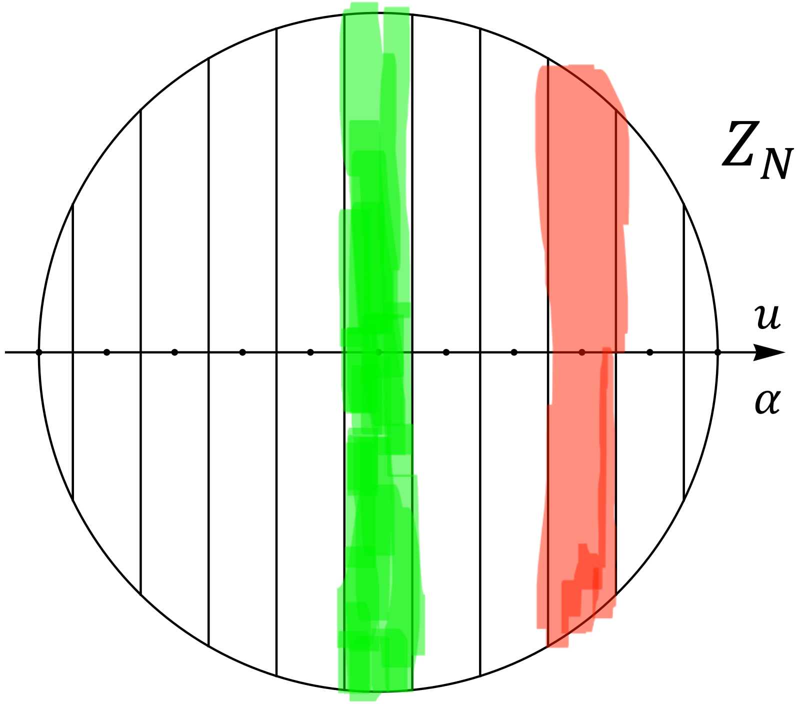
$$Z_N(D_0) \ \lessapprox \ \exp\left(N \frac{\beta^2}{2} z(1)\right)$$



$\alpha = 0$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

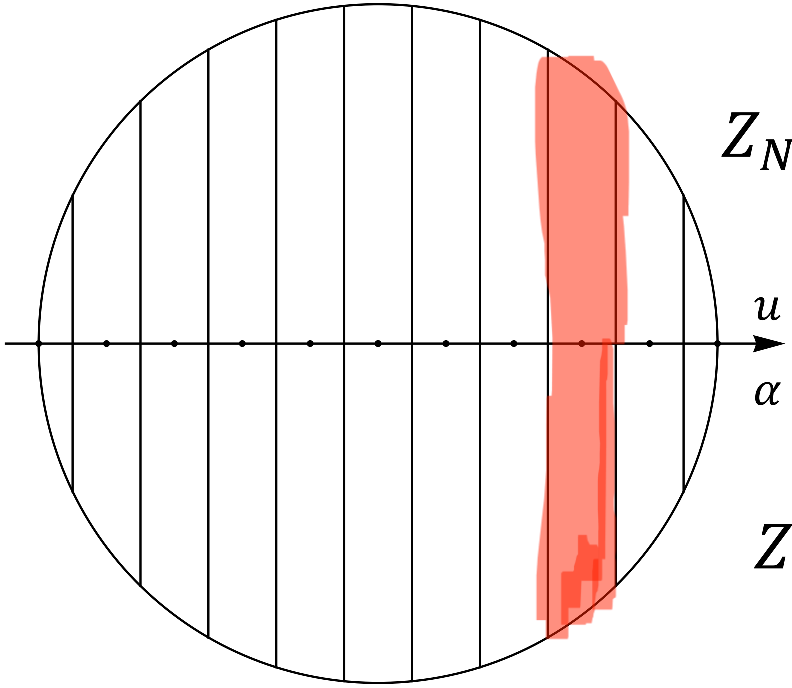
$$Z_N(D_0) \lesseqgtr \exp\left(N \frac{\beta^2}{2} z(1)\right)$$



$\alpha = 0$

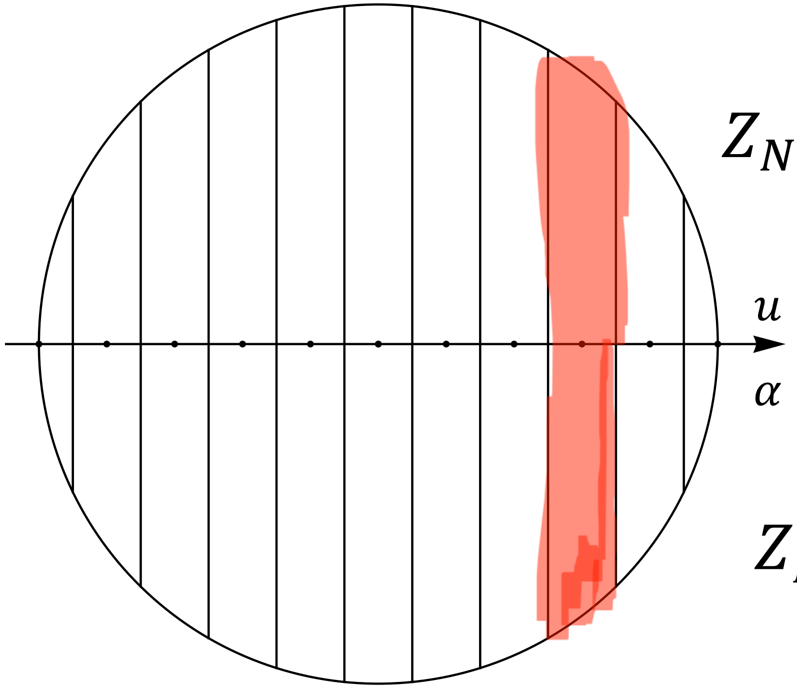
$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$



$$Z_N = \underset{\exp\left(N\frac{\beta^2}{2}z(1)\right)}{\underset{\lessapprox}{Z_N(D_0)}} + \sum_{\alpha\in A\setminus\{0\}} Z_N(D_\alpha)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$



$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \mathrel{\tilde{\leqq}} \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N(m+\hat{\sigma}) =: H_N(m) + \nabla H_N(m)\cdot\hat{\sigma} + H_N^m(\hat{\sigma})$$

$$H_N(m) \qquad (\nabla H_N(m)\cdot\hat{\sigma})_{\hat{\sigma}:\hat{\sigma}\cdot m=0} \qquad \left(H_N^m(\hat{\sigma})\right)_{\hat{\sigma}:\hat{\sigma}\cdot m=0}$$

Independent Gaussian processes!

$$\left(H_N^m(\hat{\sigma})\right)_{\hat{\sigma}:\hat{\sigma}\cdot m=0} \quad \begin{cases} \mathbb{E}[H_N^m(\hat{\sigma})H_N^m(\hat{\tau})] = N z_{\alpha^2}\left(\dfrac{\hat{\sigma}\cdot\hat{\tau}}{N}\right) \\ z_q(x) = z(q+x) - z'(q)x - z(q) \end{cases}$$

$$Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N(m + \hat{\sigma}) =: H_N(m) + \nabla H_N(m) \cdot \hat{\sigma} + H_N^m(\hat{\sigma})$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$H_N(\sigma)$$

$$=$$

$$H_N(m) + \nabla H_N(m) \cdot \hat{\sigma} + H_N^m(\hat{\sigma})$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\beta H_N(\sigma)$$

$$=$$

$$\beta H_N(m) + \beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$Q_N\left[\exp\left(\beta H_N(m) + \beta \nabla H_N(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha\right]$$

$$Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha]$$

$$=$$

$$Q_N[\exp\left(\beta H_N(m) + \beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right) \,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$Q_N\left[\exp\left(\beta H_N(m) + \beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)\,|D_\alpha\right]$$

$$=$$

$$\exp\left(\beta H_N(m)\right) \times Q_N\left[\exp\left(\beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)\,|D_\alpha\right]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$\exp\left(\beta H_N(m)\right) \times Q_N[\exp\left(\beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

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$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$\exp\big(\beta H_N(m)\big) \times Q_N\big[\exp\big(\beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\big)\,|D_\alpha\big]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$\exp\big(\beta H_N(m)\big) \times Q_N[\exp\big(\beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\big)\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$Q_N[\exp\big(\beta \nabla H_N(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\big)\,|D_\alpha]$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$Q_N[\exp\left(\beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha]$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

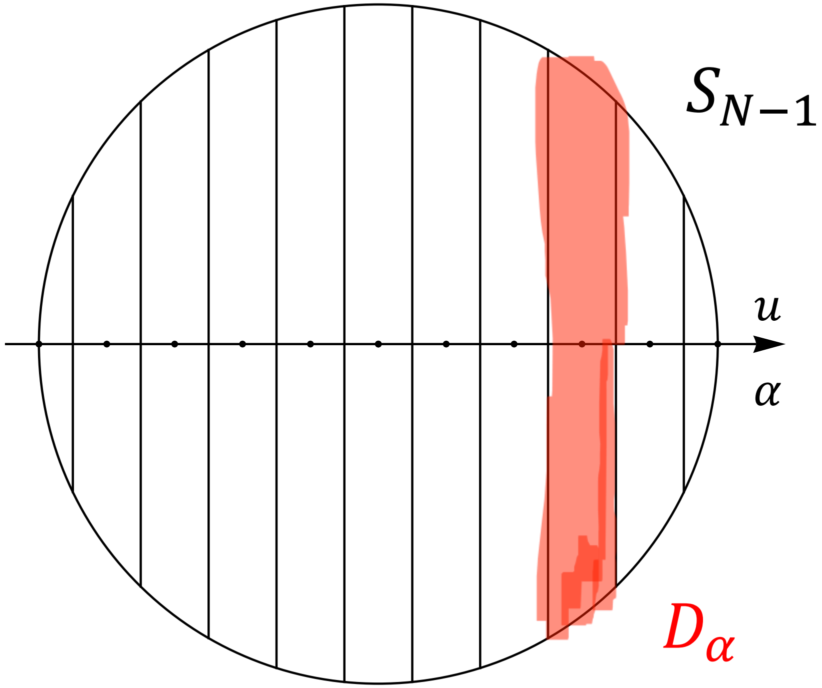
$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$Q_N[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha]$$

$$:= \nabla H_N(m)$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$



$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

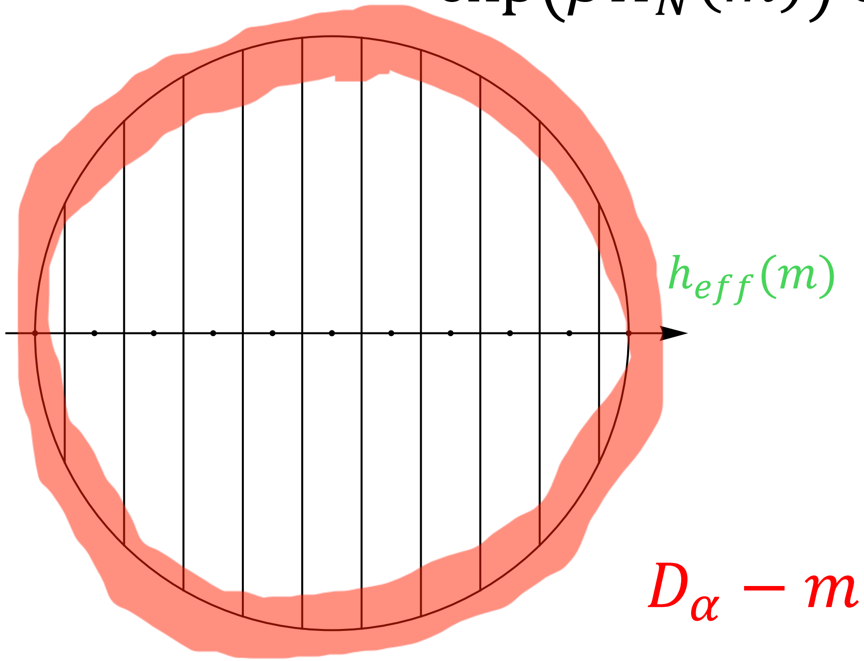
$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

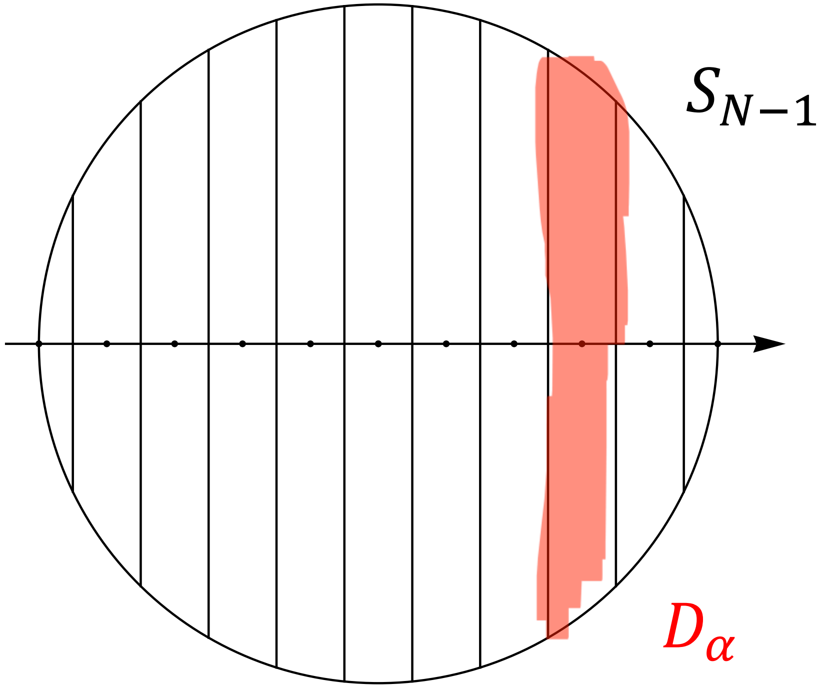
$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\left[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha\right]$$



$$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$$



$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

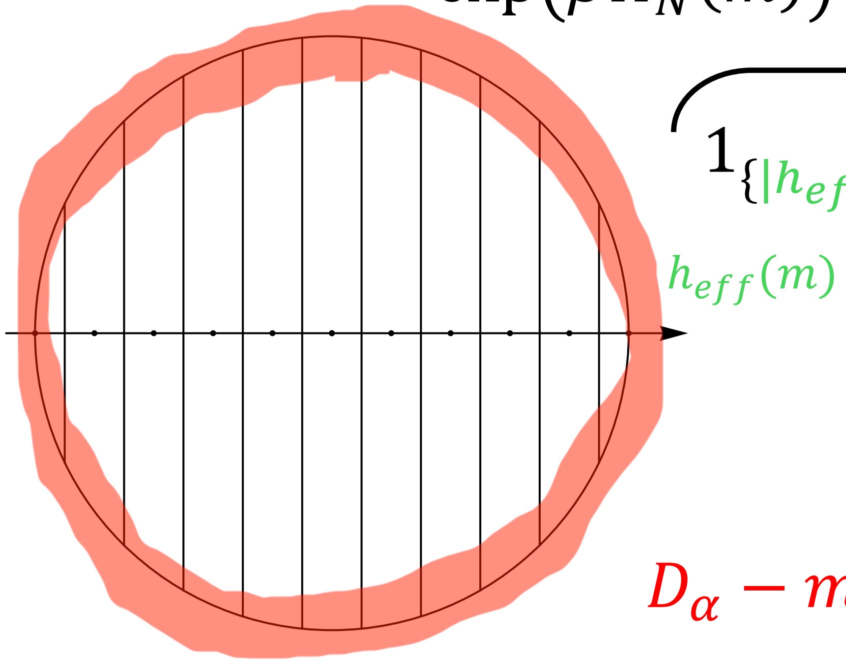
$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

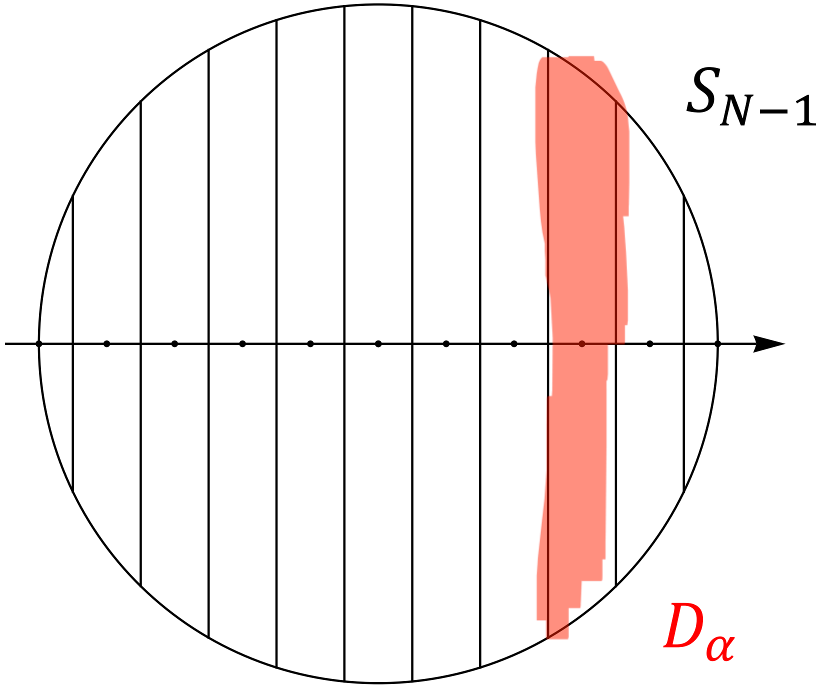
$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\Big[\underbrace{\exp}_{1_{\{|h_{eff}(m)\cdot\hat{\sigma}|\leq N^{-1/3}\}}}\Big(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\Big)\,|D_\alpha\Big]$$



$$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$$



$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

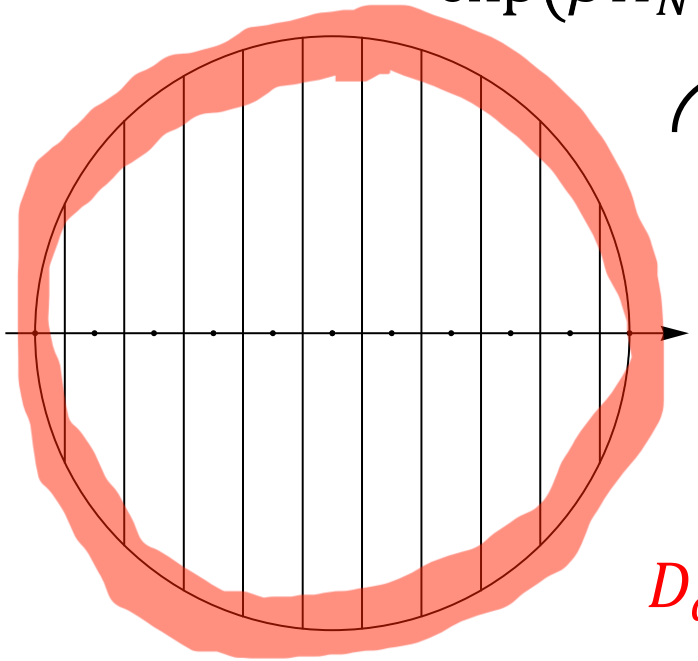
$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

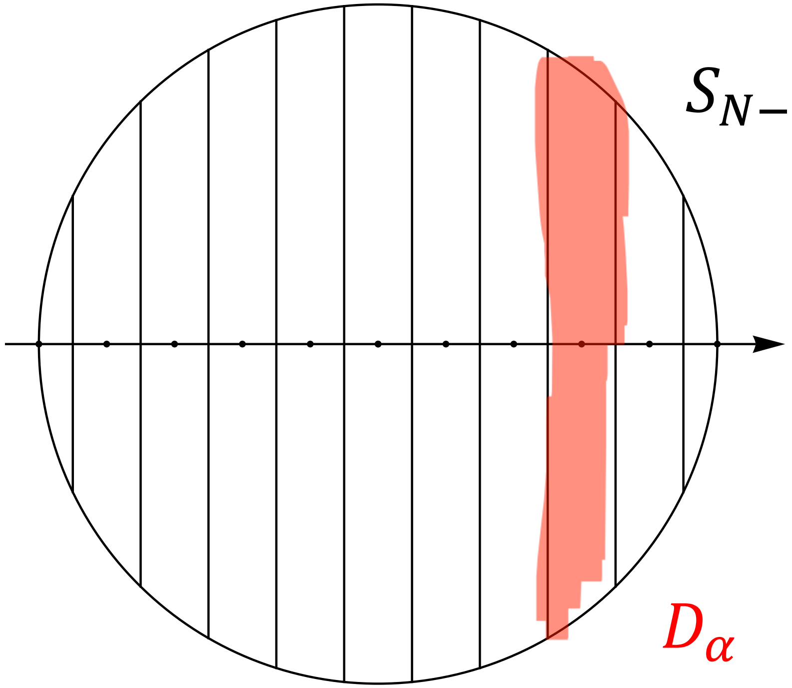
$$\exp\big(\beta H_N(m)\big)\, Q_N\Big[\underbrace{\exp}_{1_{\{|h_{eff}(m)\cdot\hat{\sigma}| \le N^{-1/3}\}}} \Big(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\Big)\,|D_\alpha\Big]$$

$$1_{\{|h_{eff}(m)\cdot\hat{\sigma}| \le N^{-1/3}\}} = 1_{\{|h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}}$$



$$E_m := D_\alpha \cap \{h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}$$

$$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$$



S_{N-1}

D_α

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

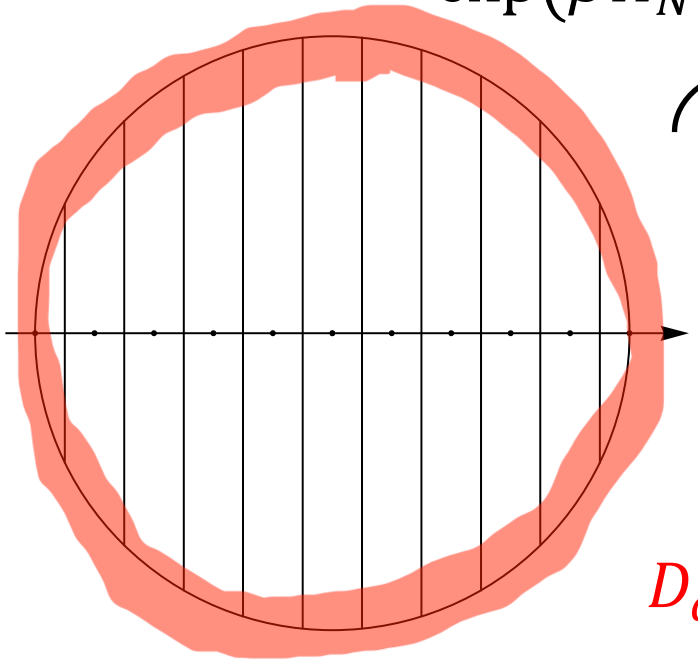
$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

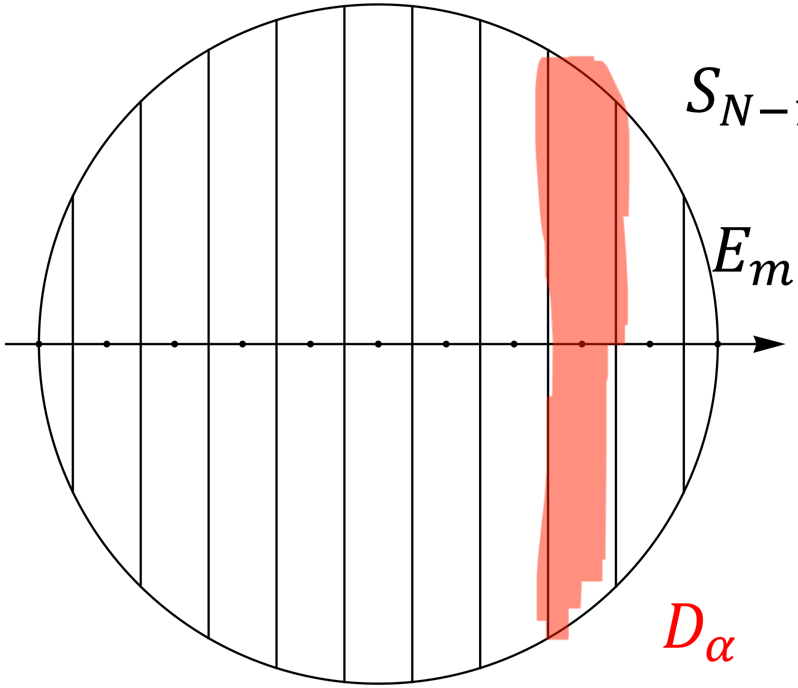
$$\exp\big(\beta H_N(m)\big)\, Q_N\left[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha\right]$$

$$1_{\{|h_{eff}(m)\cdot\hat{\sigma}|\leq N^{-1/3}\}} = 1_{\{|h_{eff}(m)\cdot(\sigma - m)|\leq N^{-1/3}\}}$$



$$E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1}\colon |h_{eff}(m)\cdot(\sigma - m)| \leq N^{-1/3}\}$$

$$D_\alpha - m \simeq S_{N-2}\big(\sqrt{N(1-\alpha^2)}\big)$$



S_{N-1}

$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$

$E_m := D_\alpha \cap \{\sigma \in S_{N-1} : |h_{eff}(m)\cdot(\sigma - m)| \leq N^{-1/3}\}$

$m = m_\alpha = \alpha u$

$\sigma =: m + \hat{\sigma}$

$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$

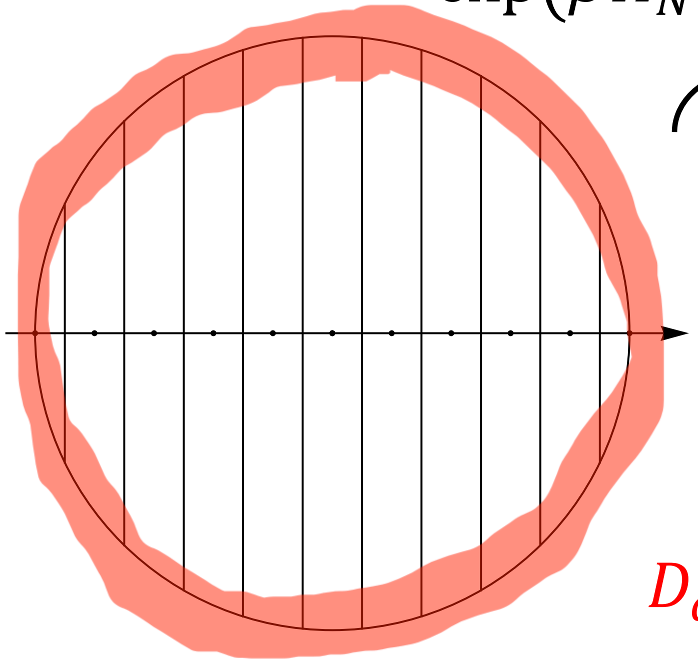
$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

D_α

$$\exp\big(\beta H_N(m)\big)\, Q_N\Big[\exp\Big(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\Big)\,|D_\alpha\Big]$$

$$1_{\{|h_{eff}(m)\cdot\hat{\sigma}|\leq N^{-1/3}\}} = 1_{\{|h_{eff}(m)\cdot(\sigma - m)|\leq N^{-1/3}\}}$$



$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m := D_\alpha \cap \{\sigma \in S_{N-1}: |h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$=$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\left[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha\right]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m := D_\alpha \cap \{\sigma \in S_{N-1}: |h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}$$

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$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m]$$

$$=$$

$$\exp(\beta H_N(m))\,Q_N[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|E_m]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m := D_\alpha \cap \{\sigma \in S_{N-1}: |h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}$$

$$m = m_\alpha = \alpha u$$

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$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m]$$

$$=$$

$$\exp(\beta H_N(m))\,Q_N[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|E_m]$$

$$\cong$$

$$\exp(\beta H_N(m))\,Q_N[\exp(\beta H_N^m(\hat{\sigma}))\,|E_m]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m := D_\alpha \cap \{\sigma \in S_{N-1}: |h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m]$$

$$=$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\Big[\exp\Big(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\Big)\,|E_m\Big]$$

$$\cong$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\big[\exp\big(\beta H_N^m(\hat{\sigma})\big)\,|E_m\big]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m := D_\alpha \cap \{\sigma \in S_{N-1} : |h_{eff}(m)\cdot(\sigma - m)| \le N^{-1/3}\}$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m]$$

$$=$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\left[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|E_m\right]$$

$$\cong$$

$$\exp\big(\beta H_N(m)\big)\, \underbrace{Q_N[\exp\big(\beta H_N^m(\hat{\sigma})\big)\,|E_m]}$$

$$\lessapprox \exp\left(N\frac{\beta^2}{2} z_{\alpha^2}(1-\alpha^2)\right)$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m := D_\alpha \cap \{\sigma \in S_{N-1}\colon |h_{eff}(m)\cdot(\sigma - m)| \leq N^{-1/3}\}$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m]$$

$$=$$

$$\exp\big(\beta H_N(m)\big)\, Q_N\left[\exp\left(\beta h_{eff}(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|E_m\right]$$

$$\cong$$

$$\exp\big(\beta H_N(m)\big)\, \underbrace{Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big)\,|E_m\right]}_{\lessapprox\ \exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right)}$$

$$Z_N = \underbrace{Z_N(D_0)}_{\substack{\lessapprox \\ \exp\left(N\frac{\beta^2}{2}z(1)\right)}} + \sum_{\alpha \in A\setminus\{0\}} Z_N(D_\alpha)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m] \cong \exp\left(\beta H_N(m)\right) \underbrace{Q_N[\exp\left(\beta H_N^m(\hat{\sigma})\right)|E_m]}_{\lessapprox\ \exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right)}$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) \,|D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma)) \,|E_m] \cong \exp\big(\beta H_N(m)\big)\, \underbrace{Q_N[\exp\big(\beta H_N^m(\hat{\sigma})\big) \,|E_m]}$$

$$\lessapprox \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1-\alpha^2)\right)$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong Q_N[\exp(\beta H_N(\sigma)) \, | E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m]$$

$$Q_N[\exp(\beta H_N(\sigma)) \, | E_m] \cong \exp\big(\beta H_N(m)\big) \, Q_N[\exp\big(\beta H_N^m(\hat{\sigma})\big) \, | E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

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$$Q_N[\exp(\beta H_N(\sigma)) \,|E_m] \cong \exp\big(\beta H_N(m)\big) \, Q_N[\exp\big(\beta H_N^m(\hat{\sigma})\big) \,|E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha\in A\setminus\{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

$$Z_N(E_m) \cong Q_N[\exp(\beta H_N(\sigma))\,|E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m] \cong \exp\big(\beta H_N(m)\big)\, Q_N[\exp\big(\beta H_N^m(\hat{\sigma})\big)\,|E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong Q_N[\exp(\beta H_N(\sigma)) \,|E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m]$$

$$Z_N \quad = \quad \underset{\exp\left(N\frac{\beta^2}{2}z(1)\right)}{\underset{\lessapprox}{Z_N(D_0)}} \quad + \quad \sum_{\alpha\in A\backslash\{0\}} Z_N(D_\alpha)$$

$$Z_N(E_m) \cong Q_N[\exp(\beta H_N(\sigma))\,|E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\underset{\exp\left(N\frac{\beta^2}{2}z(1)\right)}{\lesssim}$$

$$Z_N(E_m) \cong Q_N[\exp(\beta H_N(\sigma))\,|E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m]$$

$$\cong$$

$$\exp\left(\beta H_N(m)\right) Q_N[\exp\left(\beta H_N^m(\hat{\sigma})\right)\,|E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \mid E_m\right] \times \exp(N\beta h \alpha) \times Q_N[E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \;|E_m\right] \times \exp(N\beta h\alpha) \times Q_N[E_m]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \mid E_m\right]$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A\setminus\{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big)\ |E_m\right]$$

$$Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big)\ |E_m\right] \lessapprox \exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right)$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\lessapprox$$

$$\exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \mid E_m\right]$$

$$\lessapprox$$

$$\exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1-\alpha^2)\right)$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\lessapprox$$

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$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \,|E_m\right]$$

$$Q_N[E_m] \cong \exp\left(\frac{N}{2}\log(1-\alpha^2)\right)$$

$$Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \,|E_m\right] \lessapprox \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1-\alpha^2)\right)$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\lessapprox$$

$$\exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N\left[\exp\big(\beta H_N^m(\hat{\sigma})\big) \mid E_m\right]$$

$$\cong \qquad \cong \qquad \lessapprox$$

$$\exp(\beta h(m \cdot u)) \qquad \exp\left(\frac{N}{2}\log(1-\alpha^2)\right) \qquad \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1-\alpha^2)\right)$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp\big(\beta H_N(m)\big) \times \exp(\beta h(m \cdot u)) \times Q_N[E_m] \times Q_N\left[\exp\left(\beta H_N^m(\hat{\sigma})\right) \,|E_m\right]$$

$$Q_N[E_m] \cong \exp\left(\frac{N}{2} \log(1 - \alpha^2)\right)$$

$$Q_N\left[\exp\left(\beta H_N^m(\hat{\sigma})\right) \,|E_m\right] \lessapprox \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right)$$

$$Z_N \quad = \quad \underset{\exp\left(N\frac{\beta^2}{2}z(1)\right)}{\underset{\lesssim}{Z_N(D_0)}} \quad + \sum_{\alpha\in A\setminus\{0\}} Z_N(D_\alpha)$$

$$Z_N(E_m) \cong \exp(\beta H_N(m) + h(m\cdot u)) \times \underset{\exp\left(\frac{N}{2}\log(1-\alpha^2)\right)}{\underset{\cong}{Q_N[E_m]}} \times \underset{\exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right)}{\underset{\lesssim}{Q_N\left[\exp\left(\beta H_N^m(\hat{\sigma})\right)\ |E_m\right]}}$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lessapprox \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp(\beta H_N(m) + h(m \cdot u)) \times Q_N[E_m] \times Q_N[\exp\left(\beta H_N^m(\hat{\sigma})\right) \; | E_m]$$

$$Q_N[E_m] \cong \exp\left(\frac{N}{2} \log\left(1 - \alpha^2\right)\right)$$

$$Q_N[\exp\left(\beta H_N^m(\hat{\sigma})\right) \; | E_m] \lessapprox \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right)$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A\setminus\{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lesssim \exp\left(N\frac{\beta^2}{2}z(1)\right)$$

$$Z_N(E_m) \cong \exp(\beta H_N(m) + h(m\cdot u)) \times Q_N[E_m] \times Q_N\left[\exp\left(\beta H_N^m(\hat{\sigma})\right) \, |E_m\right]$$

$$\exp(\beta H_N(m) + h(m\cdot u)) = \exp\left(\beta H_N^h(m)\right)$$

$$Q_N[E_m] \cong \exp\left(\frac{N}{2}\log(1-\alpha^2)\right)$$

$$Q_N\left[\exp\left(\beta H_N^m(\hat{\sigma})\right) \, |E_m\right] \lesssim \exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right)$$

$$\exp\left(\beta H_N^h(m)\right)\exp\left(\frac{N}{2}\log(1-\alpha^2)\right)\exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right) \lesssim \exp\Bigg(\underbrace{\beta H_N^h(m) + \frac{N}{2}\log(1-q) + N\frac{\beta^2}{2}z_q(1-q)}_{F_{TAP}(m)}\Bigg)$$

$$q := \frac{|m|^2}{N}$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\lessapprox$$

$$\exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \lessapprox \exp(F_{TAP}(m))$$

$$F_{TAP}(m) := \beta H_N^h(m) + \frac{N}{2}\log(1-q) + N\frac{\beta^2}{2} z_q(1-q)$$

$$q := \frac{|m|^2}{N}$$

TAP: Thouless-Anderson-Palmer '77

The corresponding free energy is not easily obtained from the Bethe method, and we therefore present it as a *fait accompli*, originally derived by diagram expansion :

$$F_{\mathrm{MF}} = -\sum_{(ij)} J_{ij} m_i m_j - \tfrac{1}{2}\beta \sum_{(ij)} J_{ij}{}^2 (1-m_i{}^2)(1-m_j{}^2)$$
$$+ \tfrac{1}{2} T \sum_i [(1+m_i) \ln \tfrac{1}{2}(1+m_i) + (1-m_i) \ln \tfrac{1}{2}(1-m_i)] \qquad (13)$$

Solution of 'Solvable model of a spin glass'

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both regions, they are far from complete analyses. We also encounter some 'coincidences' which require further investigation. Details of our solutions will be given elsewhere, and we attempt here only a general description of the methods.

Keep decomposing

$$Z_N = \sum_m \underbrace{Z_N(E_m)}_{\cdot \leq \exp(F_{TAP}(m) + o(N))}$$

Theorem (B '21):

- H_N and mixed p-spin Hamiltonian
- Either
  - Spherical model (Q_N uniform on S_{N-1}), *or*
  - Ising model (Q_N uniform on $\{-1,1\}^N$)

$\Rightarrow$

$$F_N \leq \sup_m \frac{1}{N} F_{TAP}(m) + o(1)$$