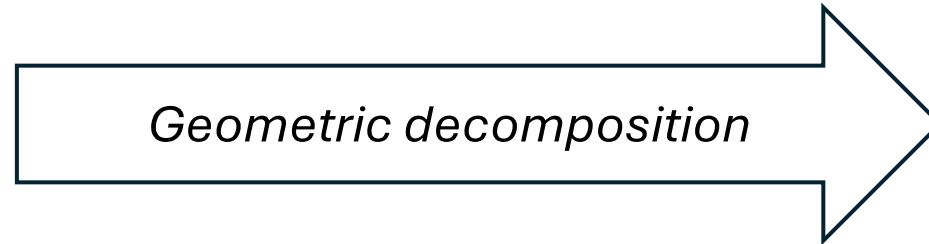


Continuation

Mixed p -spin FE when $h > 0$



TAP Free Energy
(Thouless-Anderson-Palmer)

Recap: Mixed p -spin Hamiltonian

Covariance function:
$$z(x) = \sum_{p \geq 0} a_p x^p$$

$a_p \geq 0$

Mixed p -spin Hamiltonian: H_N

- Gaussian proc. on sphere
- $\mathbb{E}[H_N(\sigma)] = 0$
- $\mathbb{E}[H_N(\sigma)H_N(\tau)] = N Z\left(\frac{\sigma \cdot \tau}{N}\right)$

Recap: Annealed Free Energy

$$\mathbb{E}(Q_N[\exp(\beta H_N(\sigma))]) = \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Q_N[\exp(\beta H_N(\sigma))] \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

Recap: Annealed Free Energy

$$\left. \begin{array}{l} a_0 \neq 0 \\ \text{or} \\ a_1 \neq 0 \end{array} \right\} \longrightarrow Q_N[\exp(\beta H_N(\sigma))] \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$\left. \begin{array}{l} a_0 = a_1 = 0 \\ \text{and} \\ \beta \leq \beta_c(z) \\ \text{and} \\ Q_N \text{ unif. on} \\ \{-1,1\}^N \text{ or } S_{N-1} \end{array} \right\} \longrightarrow Q_N[\exp(\beta H_N(\sigma))] \simeq \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

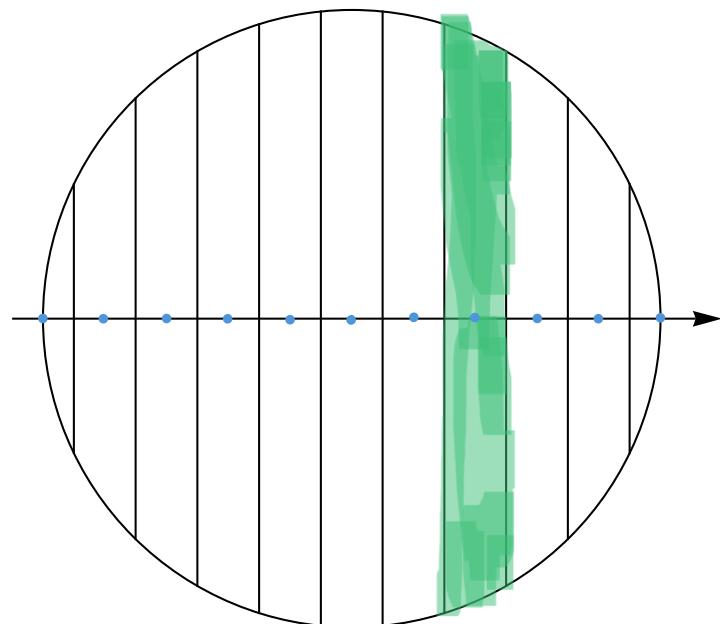
Recap: Geometric decomposition

Configuration space: S_{N-1} = sphere of radius \sqrt{N} in \mathbb{R}^N

Reference measure: Q_N = uniform prob. on S_{N-1} .

Hamiltonian: $H_N: S_{N-1} \rightarrow \mathbb{R}$

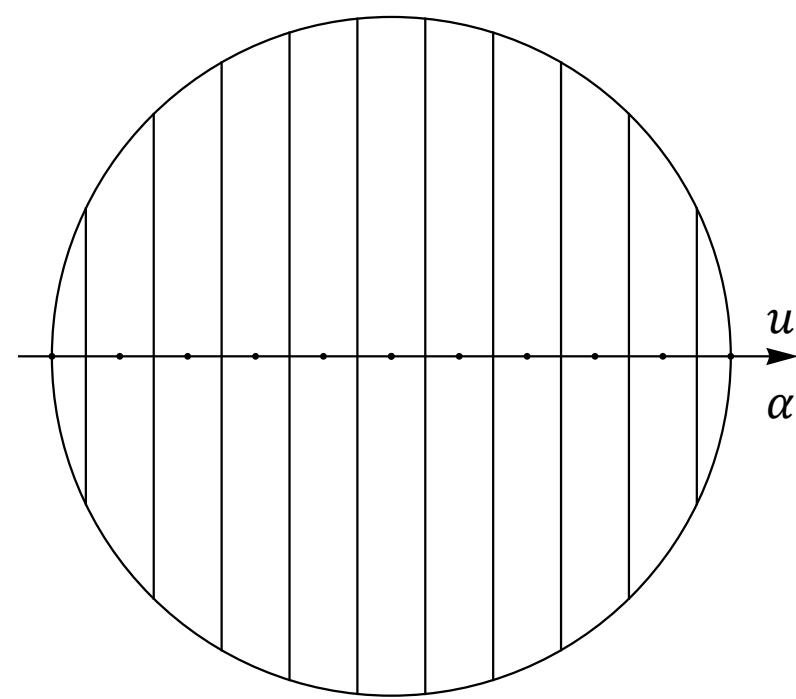
..with ext. field: $H_N^h(\sigma) = H_N(\sigma) + h(\sigma \cdot u)$



$$D_\alpha := \left\{ \sigma \in S_{N-1} : \left| \frac{\sigma \cdot u}{N} - \alpha \right| \leq N^{-1/3} \right\}$$

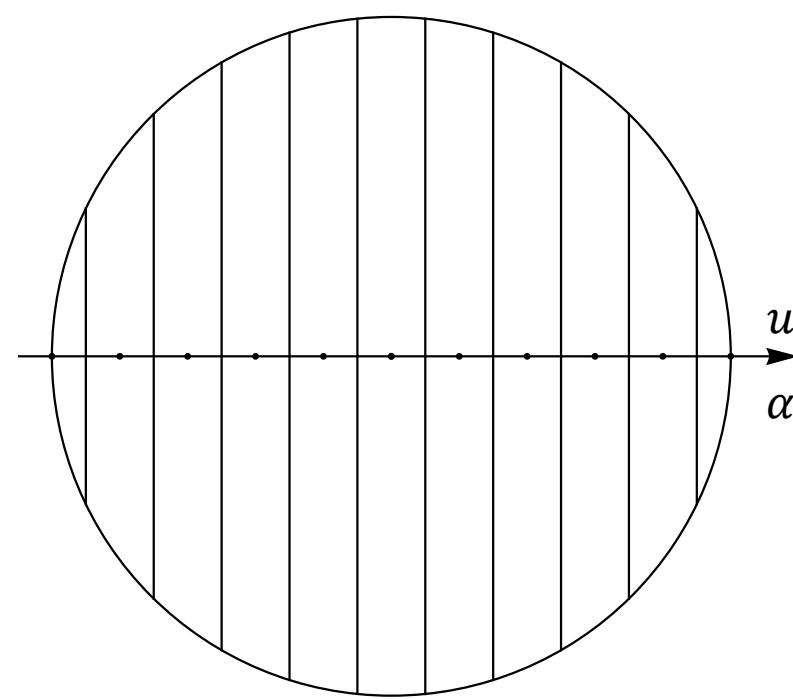
$$A := (N^{-1/3} \mathbb{Z}) \cap (-1, 1)$$

$$\begin{aligned} Z_N &= \sum_{\alpha \in A} Z_N(D_\alpha) \\ &= \sum_{\alpha \in A} Q_N[1_{D_\alpha} \exp(\beta H_N^h(\sigma))] \end{aligned}$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

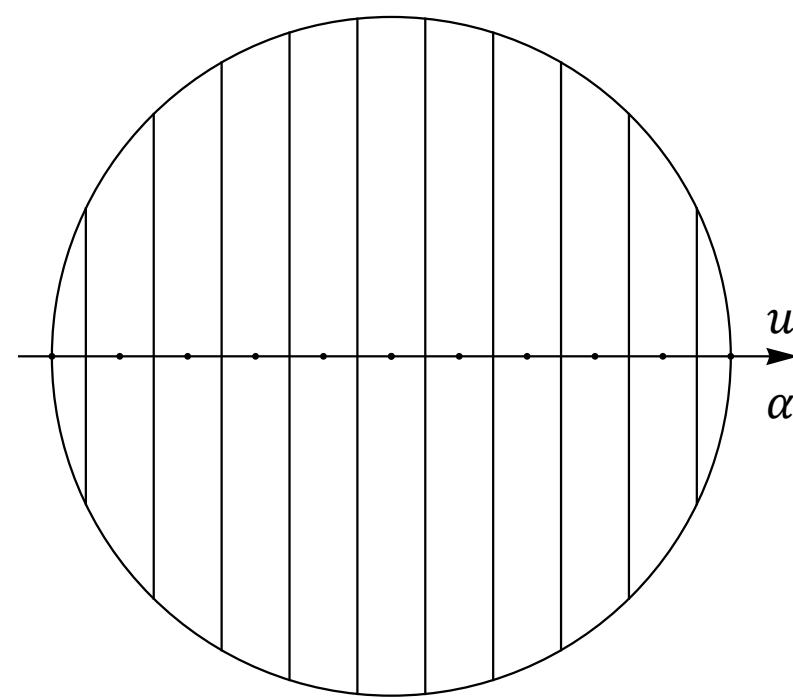


$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

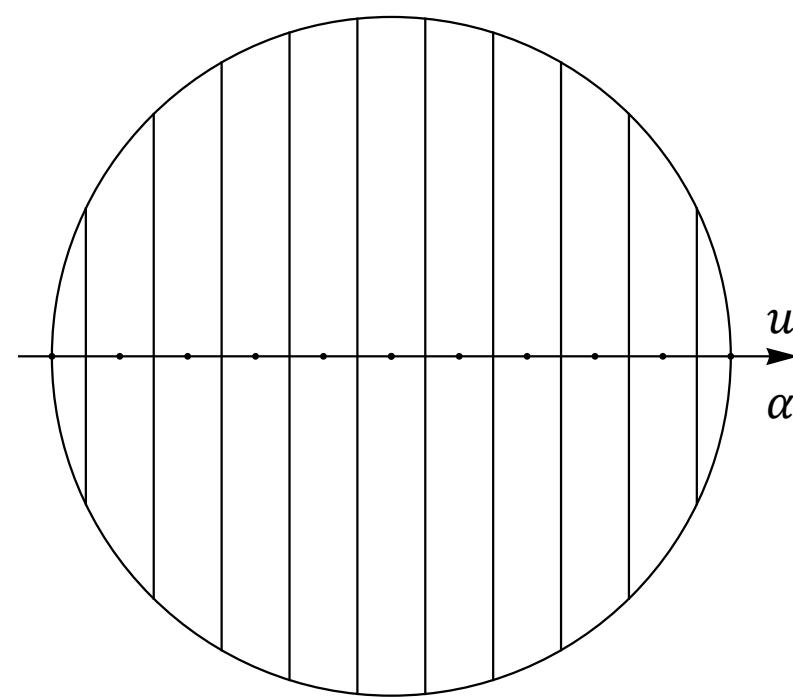
\cong

$$\exp\left(\frac{N}{2} \log(1 - \alpha^2)\right)$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

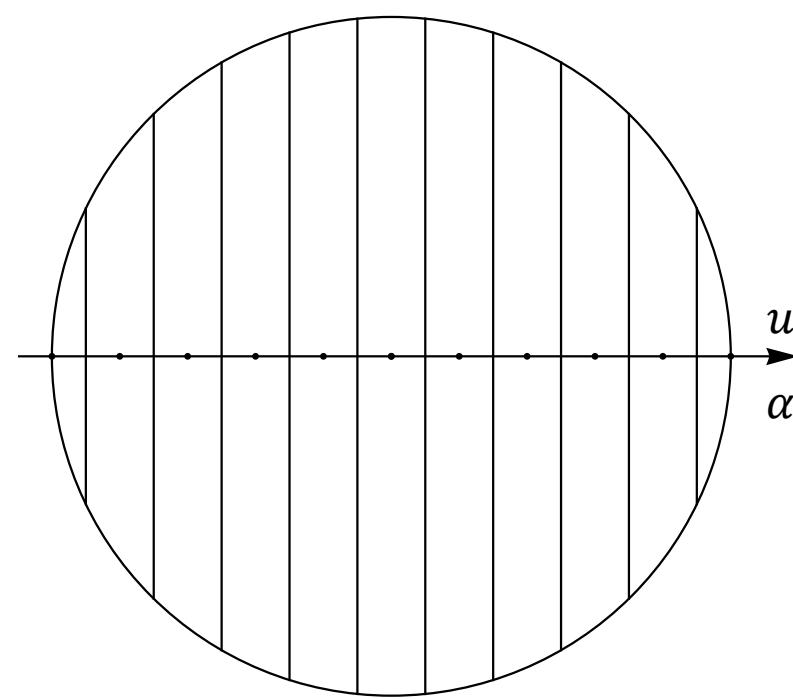
$$\begin{aligned} Z_N(D_\alpha) &\cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha] \\ &\cong \exp\left(\frac{N}{2} \log(1 - \alpha^2)\right) \end{aligned}$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Curie-Weiss

$$\begin{aligned} Z_N(D_\alpha) &\cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha] \\ &= \\ &\exp(N\beta\alpha^2) \\ &\cong \\ &\exp\left(\frac{N}{2}\log(1 - \alpha^2)\right) \end{aligned}$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Curie-Weiss

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

=

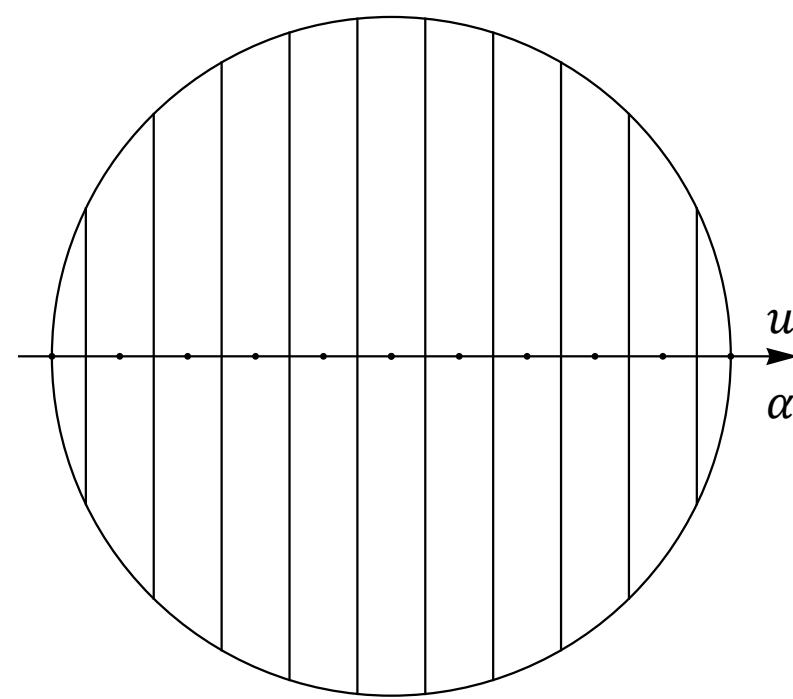
$$\exp(N\beta\alpha^2) \qquad \qquad \qquad \exp\left(\frac{N}{2}\log(1-\alpha^2)\right)$$

$$Z_N \cong \sum_{\alpha \in A} \exp(NF(\alpha))$$

$$F_N \rightarrow \sup_{\alpha \in (-1,1)} F(\alpha)$$

$$\exp\left(N \left\{ \beta\alpha^2 + \beta h\alpha + \frac{1}{2}\log(1-\alpha^2) \right\} \right)$$

$$F(\alpha)$$



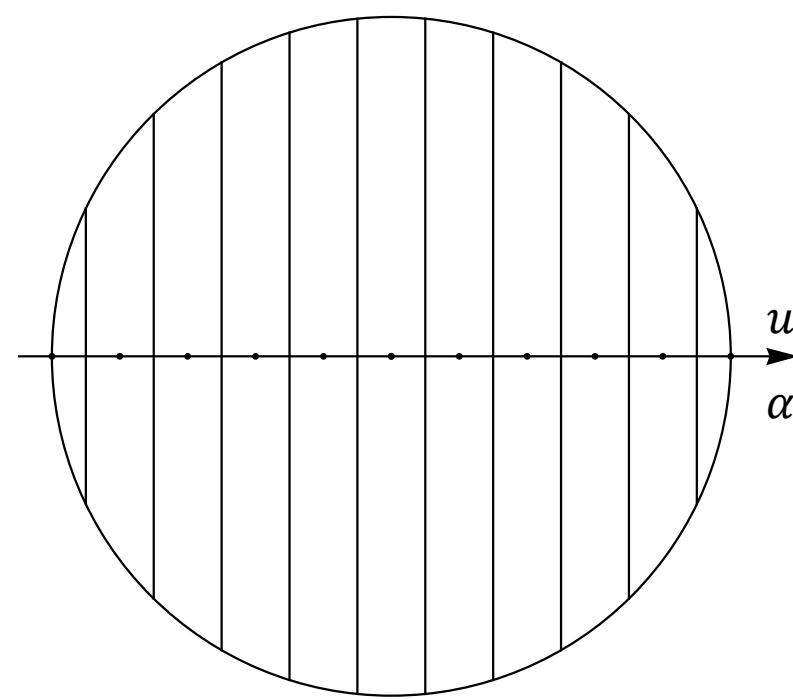
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Mixed p -spin covar $z(x)$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

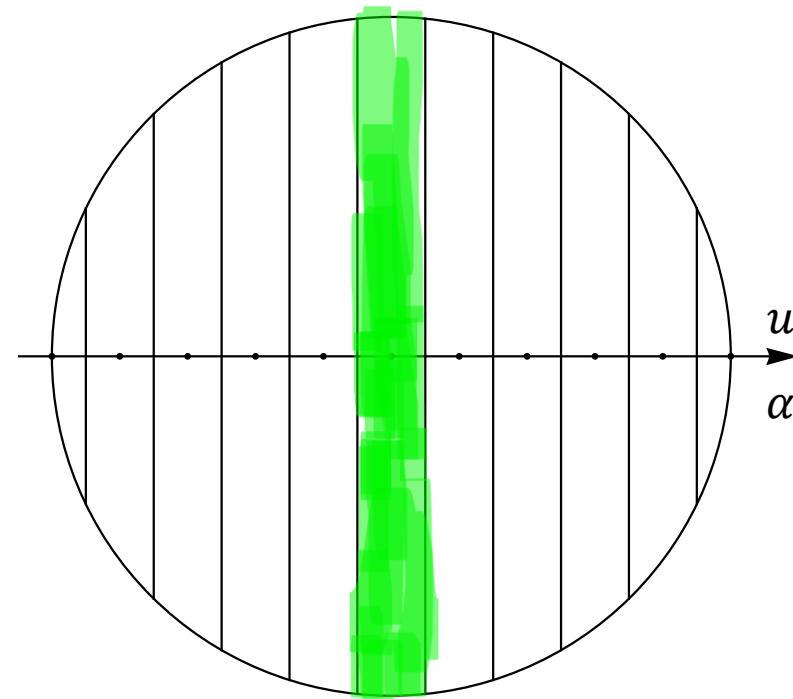
\cong

$$\exp\left(\frac{N}{2} \log(1 - \alpha^2)\right)$$



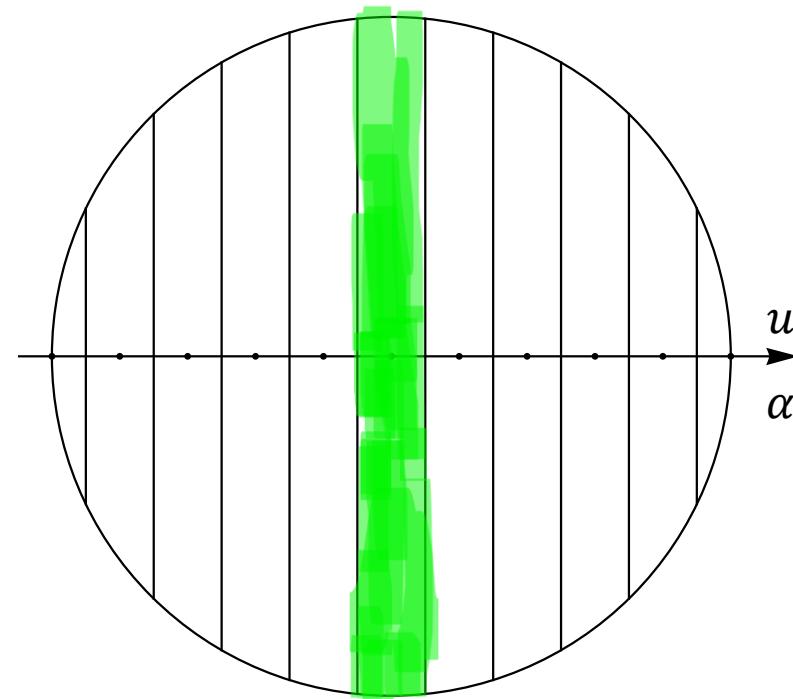
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

$$Z_N(D_{\textcolor{red}{\alpha}}) \cong Q_N[\exp(\beta H_N(\sigma)) | D_{\textcolor{red}{\alpha}}] \times \exp(N\beta h \textcolor{red}{\alpha}) \times Q_N[D_{\textcolor{red}{\alpha}}]$$



$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

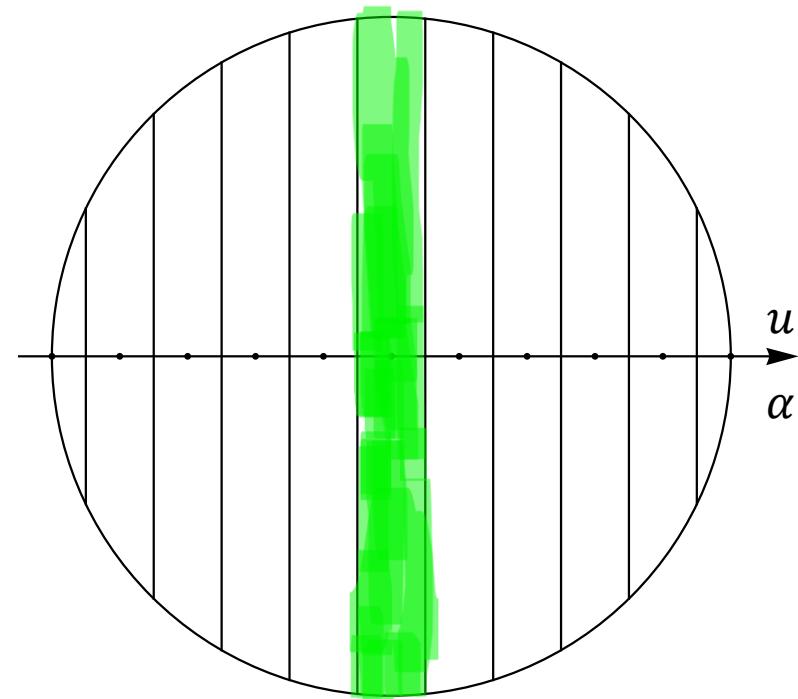
$$Z_N(D_0) \cong Q_N[\exp(\beta H_N(\sigma)) | D_0] \times \exp(N\beta h \cdot \mathbf{0}) \times Q_N[D_0]$$



$$\alpha = 0$$

$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

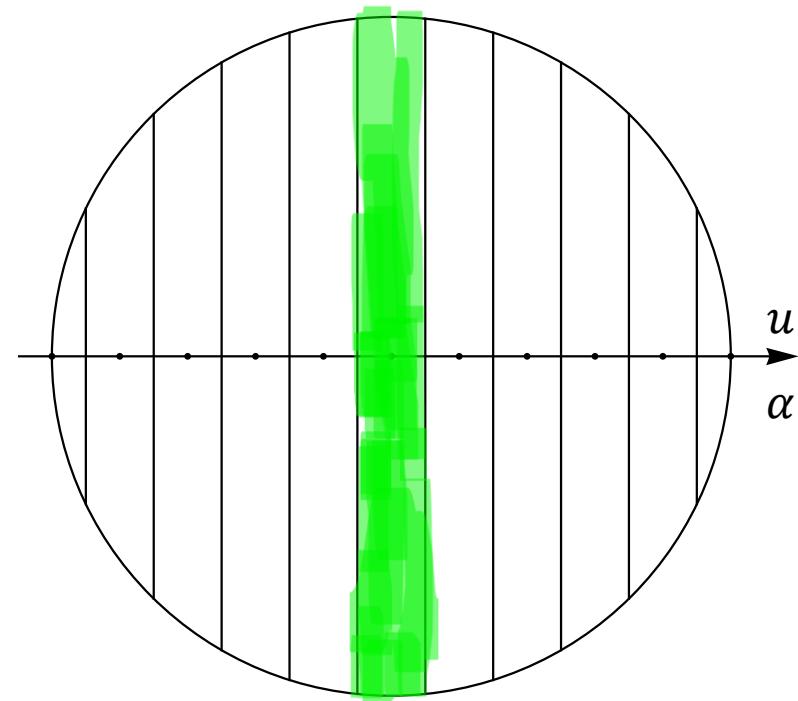
$$Z_N(D_0) \cong Q_N[\exp(\beta H_N(\sigma)) | D_0] \times \underbrace{\exp(N\beta h \cdot 0)}_{=1} \times \underbrace{Q_N[D_0]}_{\cong 1}$$



$$\alpha = 0$$

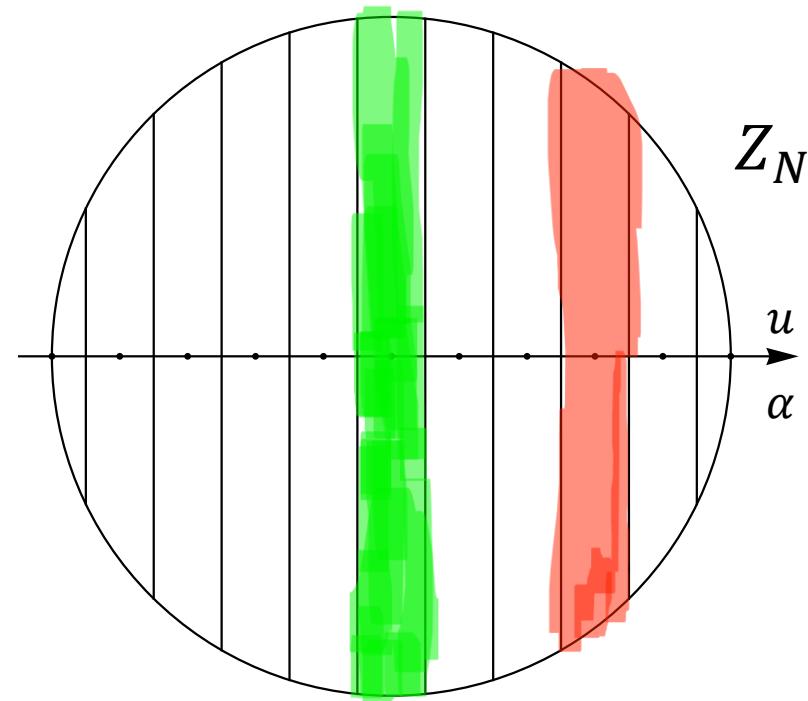
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$

$$Z_N(D_0) \cong Q_N[\exp(\beta H_N(\sigma)) | D_0] \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$



$$\alpha = 0$$

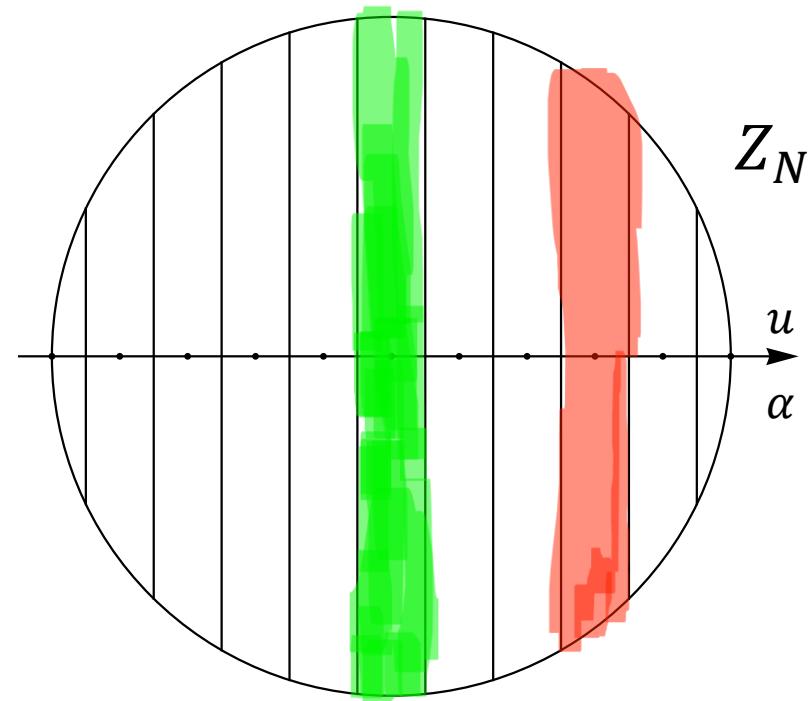
$$Z_N = \sum_{\alpha \in A} Z_N(D_\alpha)$$
$$Z_N(D_0) \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$



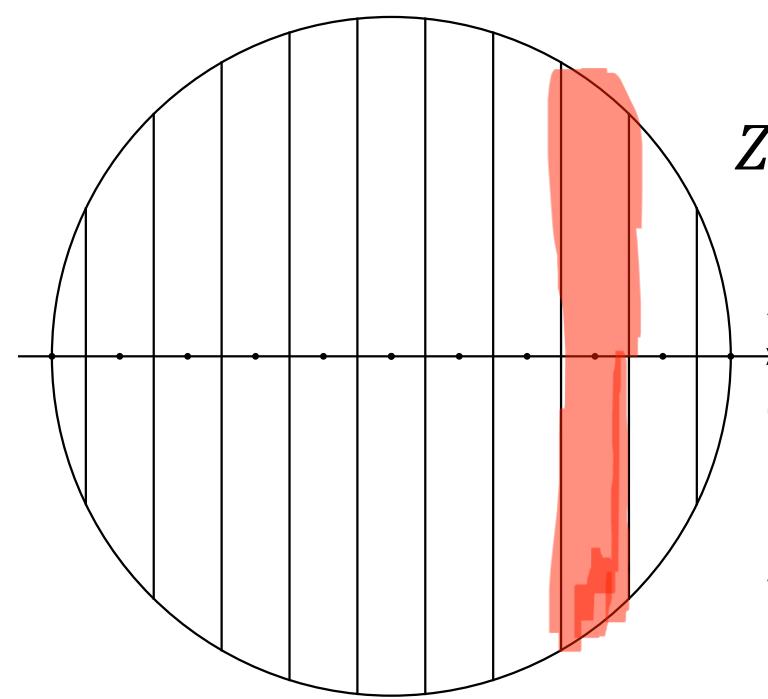
$$\alpha = 0$$

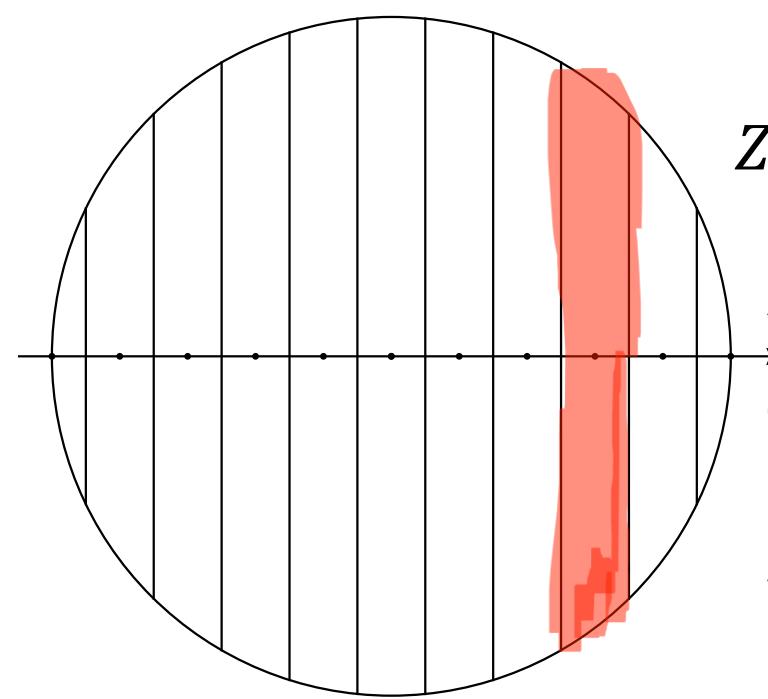
$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$Z_N(D_0) \lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$



$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$
$$\lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$


$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(\textcolor{red}{D}_\alpha)$$
$$\lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$
$$Z_N(\textcolor{red}{D}_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | \textcolor{red}{D}_\alpha] \times \exp(N\beta h\alpha) \times Q_N[\textcolor{red}{D}_\alpha]$$


$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$
$$\lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$
$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma)) \mid D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N(m + \hat{\sigma}) =: H_N(m) + \nabla H_N(m) \cdot \hat{\sigma} + H_N^m(\hat{\sigma})$$

$$\underbrace{H_N(m) \quad (\nabla H_N(m) \cdot \hat{\sigma})_{\hat{\sigma}: \hat{\sigma} \cdot m = 0} \quad (H_N^m(\hat{\sigma}))_{\hat{\sigma}: \hat{\sigma} \cdot m = 0}}_{\text{Independent Gaussian processes!}}$$

$$(H_N^m(\hat{\sigma}))_{\hat{\sigma}: \hat{\sigma} \cdot m = 0} \left\{ \begin{array}{l} \mathbb{E}[H_N^m(\hat{\sigma})H_N^m(\hat{\tau})] = Nz_{\alpha^2}\left(\frac{\hat{\sigma} \cdot \hat{\tau}}{N}\right) \\ z_q(x) = z(q+x) - z'(q)x - z(q) \end{array} \right.$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H_N(m+\hat{\sigma})=:H_N(m)+\nabla H_N(m)\cdot\hat{\sigma}+H^m_N(\hat{\sigma})$$

$$H^m_N(\hat{\sigma})\sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\left|D_\alpha\right]$$

$$m=m_\alpha=\alpha u$$

$$\textcolor{red}{H_N(\sigma)} \qquad \qquad \qquad \sigma =: m + \hat{\sigma}$$

$$= \qquad \qquad \qquad H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\textcolor{red}{H_N(m) + \nabla H_N(m) \cdot \hat{\sigma} + H^m_N(\hat{\sigma})}$$

$$Q_N[\exp(\beta H_N(\sigma))\left|D_\alpha\right]$$

$$m=m_\alpha=\alpha u$$

$$\color{red}{\beta H_N(\sigma)} \qquad \qquad \qquad \sigma =: m + \hat{\sigma}$$

$$= \qquad \qquad \qquad H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\color{red}{\beta H_N(m) + \beta \nabla H_N(m) \cdot \hat{\sigma} + \beta H^m_N(\hat{\sigma})}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$Q_N[\exp(\textcolor{red}{\beta} H_N(\sigma))\,|D_\alpha] \qquad \qquad \qquad \sigma=:m+\hat{\sigma}$$

$$\; = \;$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp\bigl(\textcolor{red}{\beta} H_N(m) + \beta\nabla H_N(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\bigr)\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$\; = \;$$

$$\sigma=:m+\hat{\sigma}$$

$$Q_N[\exp\bigl(\textcolor{red}{\beta H_N(m)} + \textcolor{green}{\beta\nabla H_N(m)\cdot\hat{\sigma}} + \textcolor{green}{\beta H_N^m(\hat{\sigma})}\bigr)\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$\; = \;$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp\bigl(\textcolor{red}{\beta H_N(m)}+\textcolor{green}{\beta\nabla H_N(m)\cdot\hat{\sigma}}+\textcolor{green}{\beta H_N^m(\hat{\sigma})}\bigr)\,|D_\alpha]$$

$$\; = \;$$

$$\exp\bigl(\textcolor{red}{\beta H_N(m)}\bigr) \times Q_N[\exp\bigl(\textcolor{green}{\beta\nabla H_N(m)\cdot\hat{\sigma}}+\textcolor{green}{\beta H_N^m(\hat{\sigma})}\bigr)\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$\sigma =: m + \hat{\sigma}$$

$$H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\quad\quad\quad =$$

$$\exp\bigl(\textcolor{red}{\beta H_N(m)}\bigr)\times Q_N[\exp\bigl(\textcolor{green}{\beta\nabla H_N(m)\cdot\hat{\sigma}}+\beta H^m_N(\hat{\sigma})\bigr)\,|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \qquad \qquad \qquad \sigma=:m+\hat{\sigma}$$

$$= \qquad \qquad \qquad H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp\!\left(\textcolor{red}{\beta H_N(m)}\right) \times Q_N[\exp\!\left(\textcolor{green}{\beta\nabla H_N(m)\cdot\hat{\sigma} + \beta H^m_N(\hat{\sigma})}\right)|D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha] \qquad \qquad \qquad \sigma=:m+\hat{\sigma}$$

$$= \qquad \qquad \qquad H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp\bigl(\beta H_N(m)\bigr) \times \textcolor{blue}{Q_N[\exp\bigl(\beta\nabla H_N(m)\cdot\hat{\sigma} + \beta H^m_N(\hat{\sigma})\bigr)\,|D_\alpha]}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$\sigma=:m+\hat{\sigma}$$

$$Q_N[\exp\bigl(\beta\nabla H_N(m)\cdot\hat\sigma + \beta H_N^m(\hat\sigma)\bigr)\,|D_\alpha]$$

$$H_N^m(\hat\sigma) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_\alpha=\alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$Q_N[\exp\bigl(\beta \textcolor{red}{\nabla H_N(m)}\cdot\hat\sigma+\beta H^m_N(\hat\sigma)\bigr)\,|D_\alpha]$$

$$H^m_N(\hat\sigma) \sim z_{\alpha^2}$$

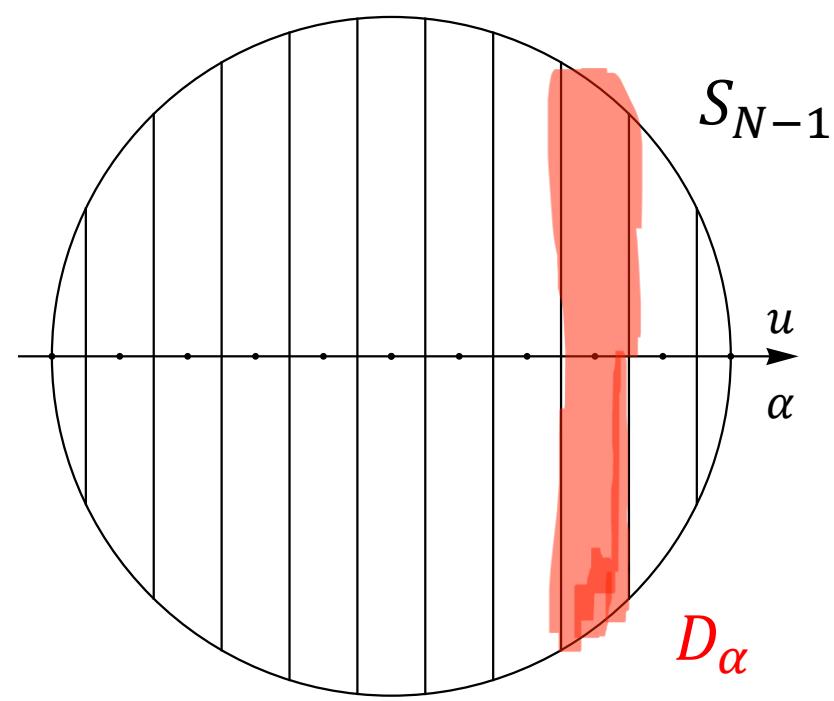
$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$m=m_{\alpha}=\alpha u$$

$$\sigma =: m + \hat{\sigma}$$

$$H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\begin{aligned} Q_N[\exp\left(\beta \textcolor{red}{h_{eff}(m)}\cdot \hat{\sigma}+\beta H_N^m(\hat{\sigma})\right)|D_\alpha]\\ :=&\textcolor{red}{\nabla H_N(m)} \end{aligned}$$



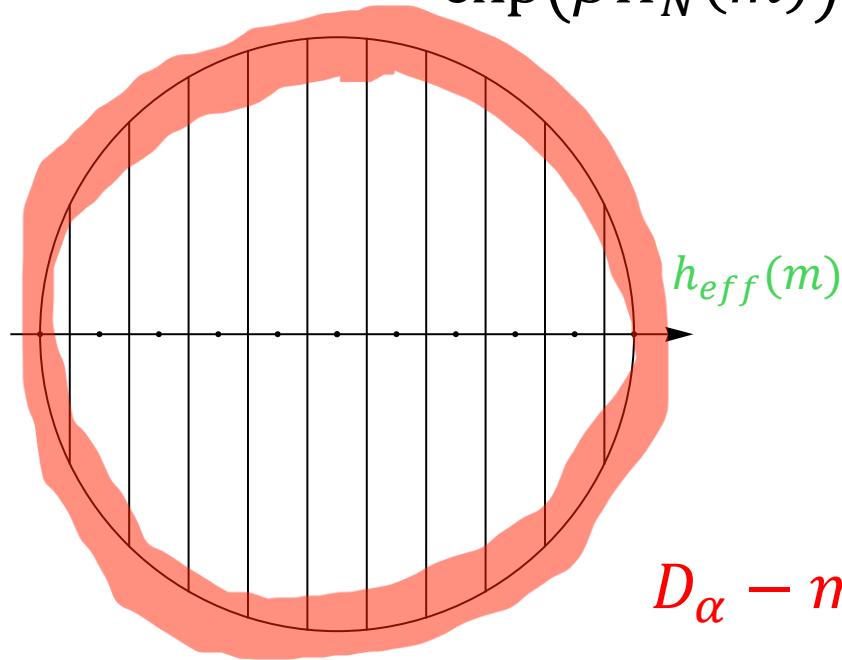
$$Q_N[\exp(\beta H_N(\sigma)) | D_\alpha]$$

$$m = m_\alpha = \alpha u$$

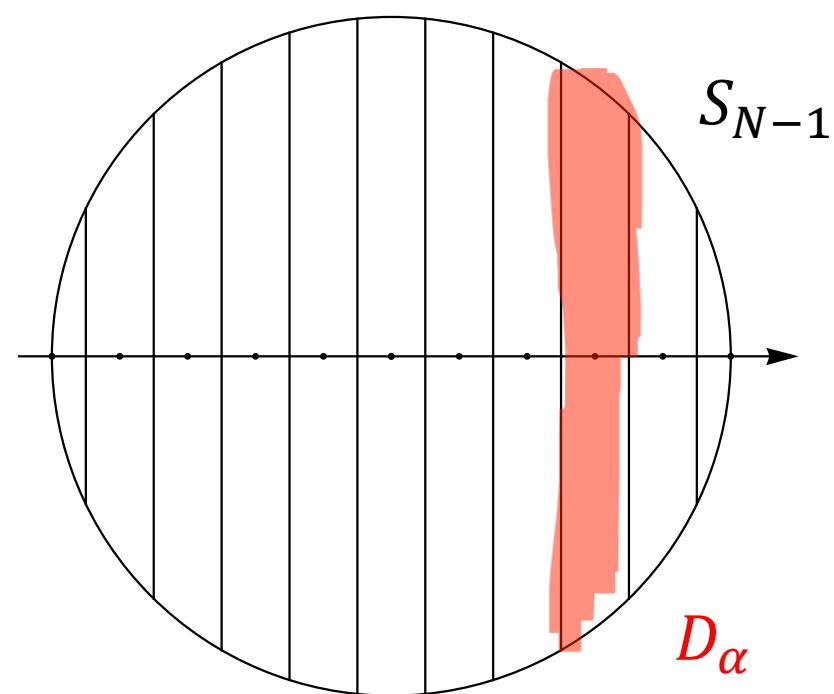
$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp(\beta H_N(m)) Q_N[\exp(\beta \textcolor{green}{h}_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})) | D_\alpha]$$



$$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$$



$$Q_N[\exp(\beta H_N(\sigma)) | D_\alpha]$$

$$m = m_\alpha = \alpha u$$

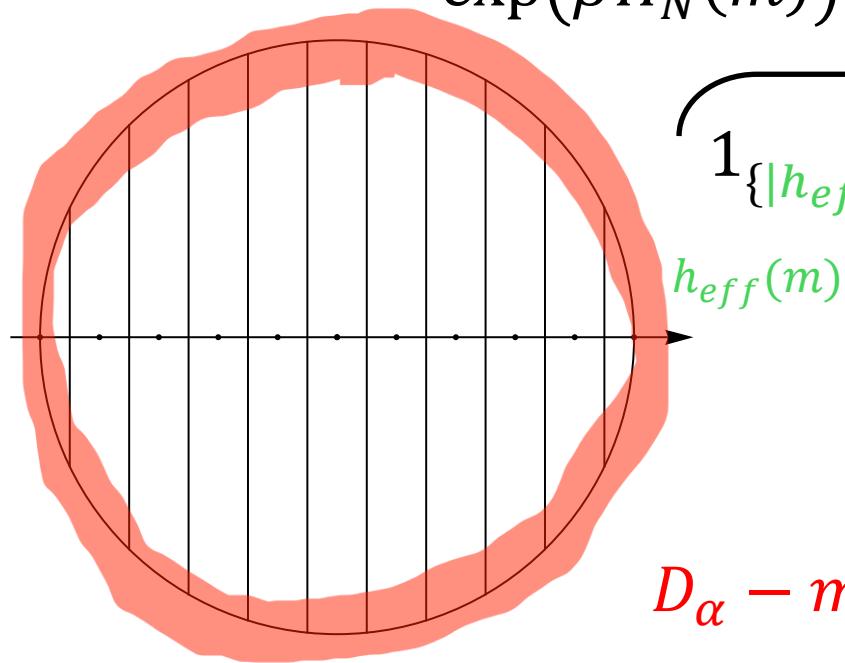
$$\sigma =: m + \hat{\sigma}$$

$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$Q_N[\exp(\beta H_N(\sigma)) | D_\alpha]$$

=

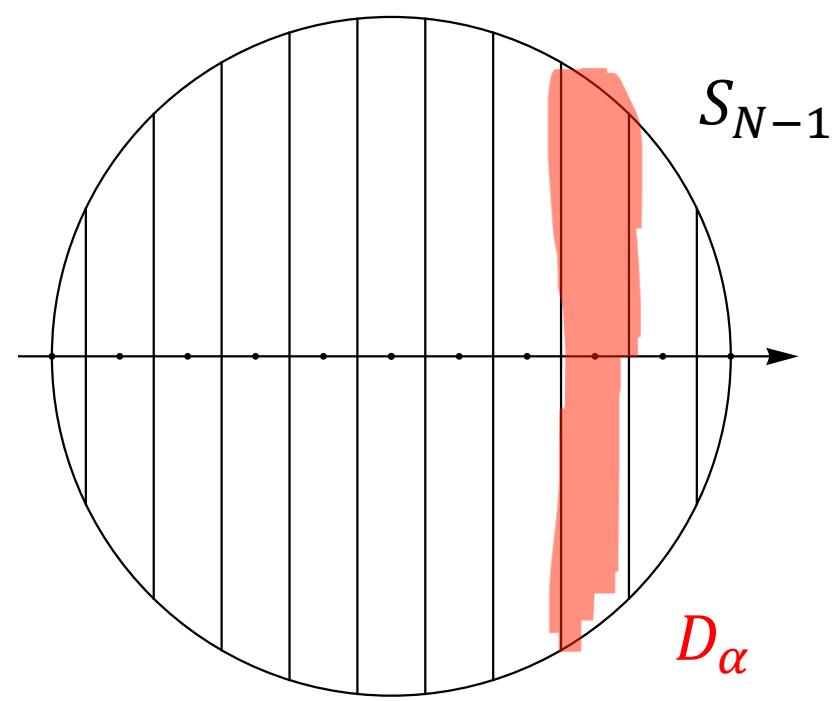
$$\exp(\beta H_N(m)) Q_N[\exp \left(\beta \textcolor{green}{h}_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma}) \right) | D_\alpha]$$



$$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$$

$$1_{\{|h_{eff}(m) \cdot \hat{\sigma}| \leq N^{-1/3}\}}$$

$$h_{eff}(m)$$



$$Q_N[\exp(\beta H_N(\sigma)) | D_\alpha]$$

$$m = m_\alpha = \alpha u$$

$$\sigma =: m + \hat{\sigma}$$

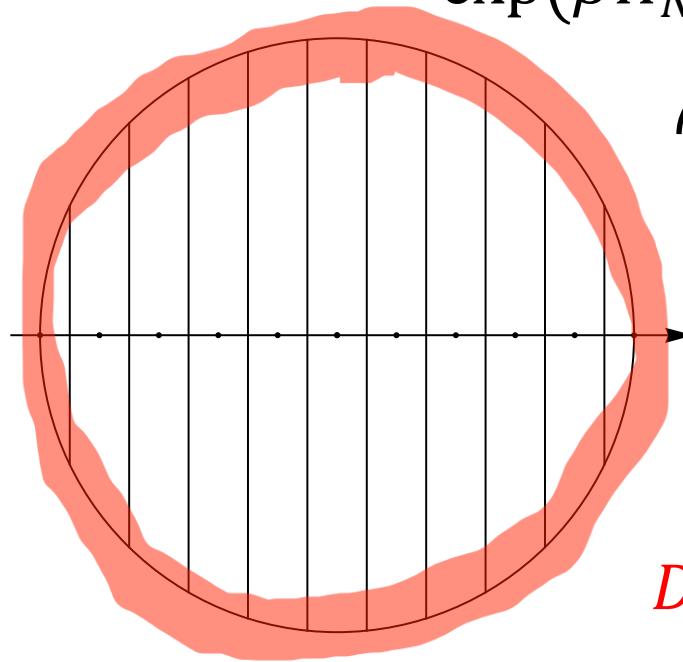
$$H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

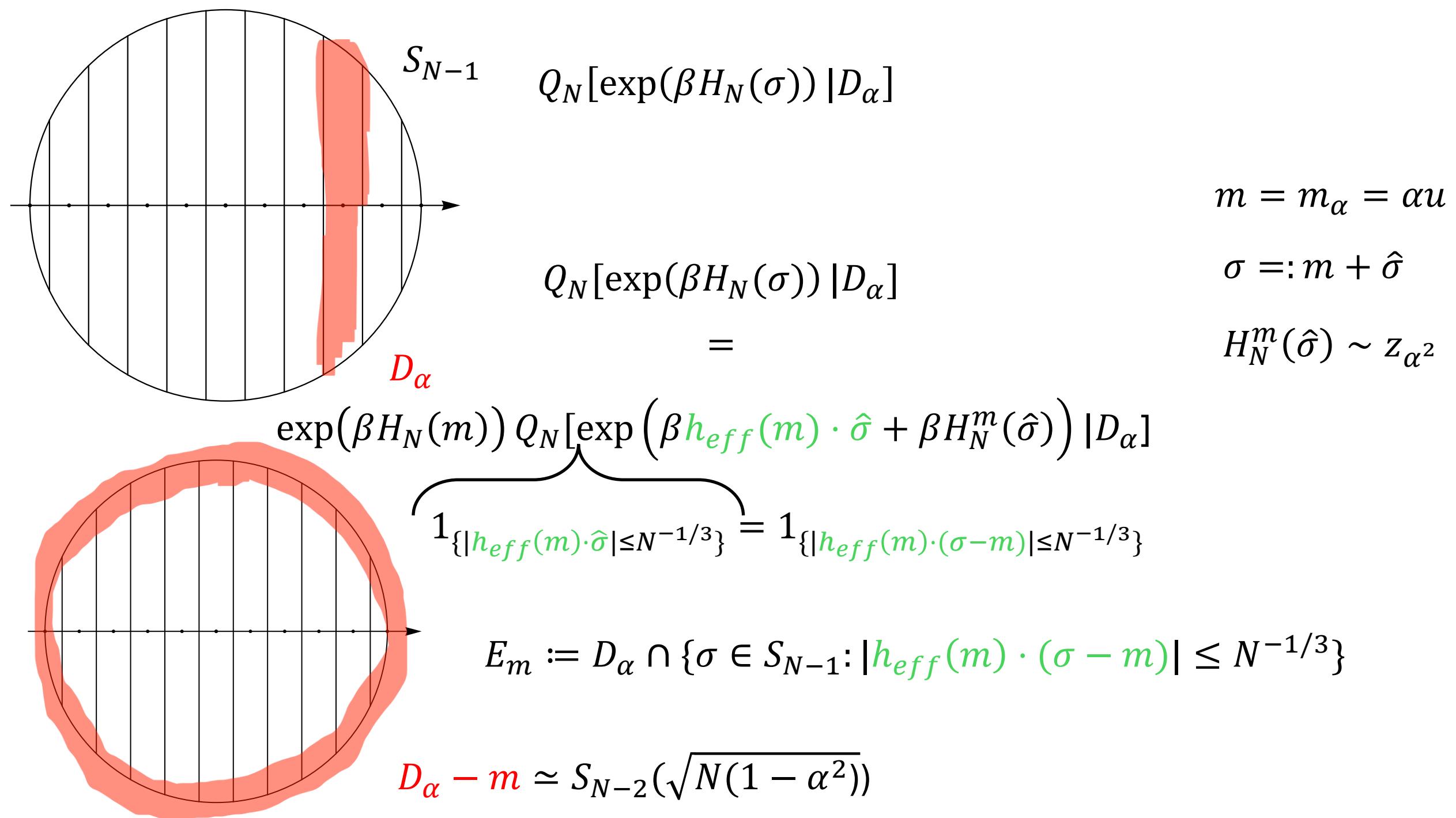
$$\exp(\beta H_N(m)) Q_N[\exp\left(\beta \textcolor{green}{h}_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right) | D_\alpha]$$

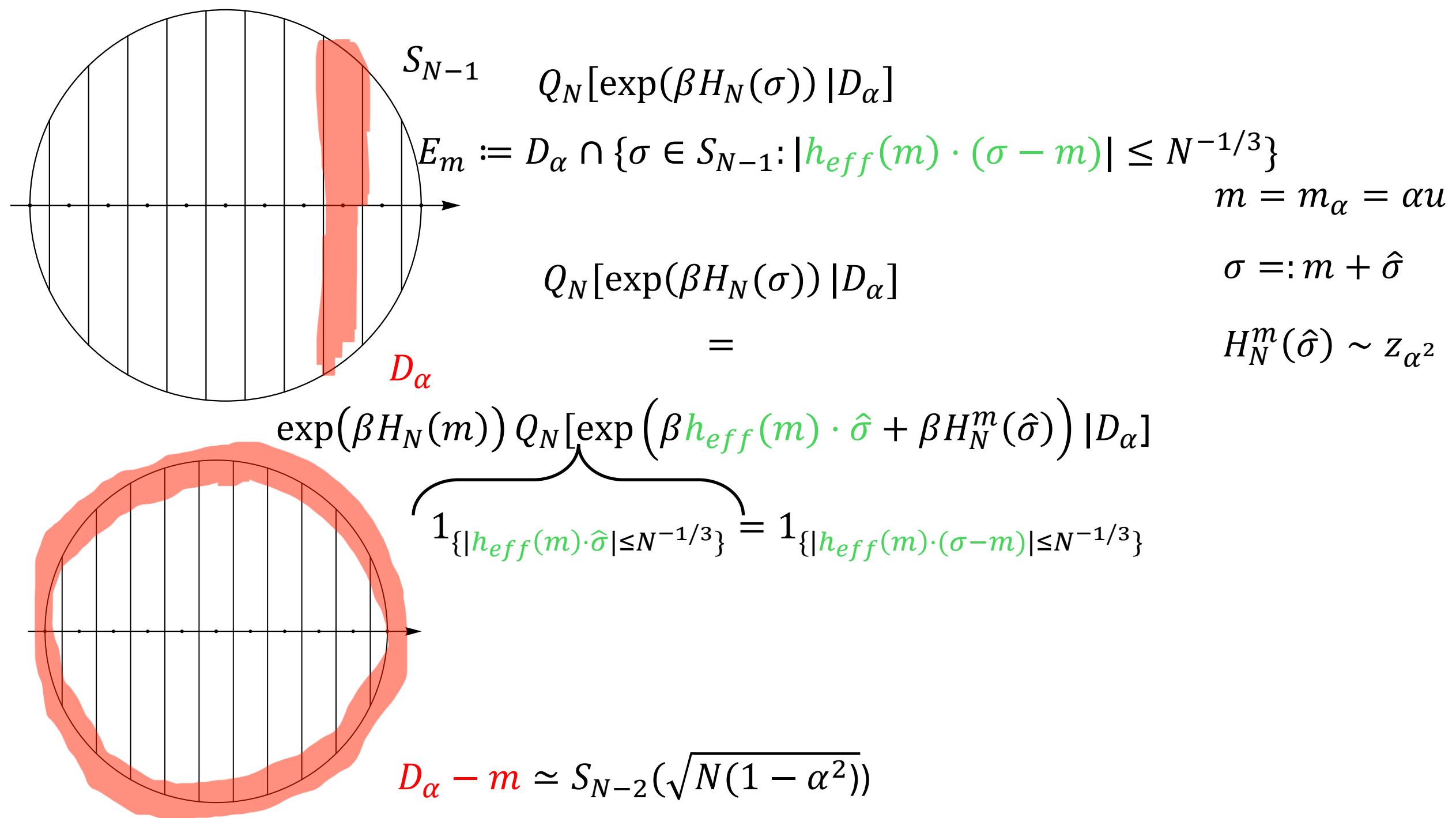
$$1_{\{|\textcolor{green}{h}_{eff}(m) \cdot \hat{\sigma}| \leq N^{-1/3}\}} = 1_{\{|\textcolor{green}{h}_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\}}$$

$$E_m := D_\alpha \cap \{|\textcolor{green}{h}_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\}$$

$$D_\alpha - m \simeq S_{N-2}(\sqrt{N(1-\alpha^2)})$$







$$Q_N[\exp(\beta H_N(\sigma))\,|\mathcal{D}_\alpha]$$

$$\begin{aligned}E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1} \colon |\textcolor{violet}{h}_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\} \\ m = m_\alpha = \alpha u\end{aligned}$$

$$\begin{aligned}Q_N[\exp(\beta H_N(\sigma))\,|\textcolor{brown}{D}_{\alpha}] \\ = \\ \sigma =: m + \hat{\sigma} \\ H_N^m(\hat{\sigma}) \sim z_{\alpha^2}\end{aligned}$$

$$\exp\bigl(\beta H_N(m)\bigr)\, Q_N[\exp\left(\beta \textcolor{violet}{h}_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)\,|\textcolor{brown}{D}_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1} \colon |\textcolor{violet}{h}_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\} \quad m = m_\alpha = \alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|\textcolor{brown}{E}_{\textcolor{blue}{m}}] \qquad \qquad \qquad \sigma =: m + \hat{\sigma}$$

$$= \qquad \qquad \qquad H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp\bigl(\beta H_N(m)\bigr)\, Q_N[\exp\left(\beta \textcolor{violet}{h}_{eff}(m) \cdot \hat{\sigma} + \beta H^m_N(\hat{\sigma})\right) | \textcolor{brown}{E}_{\textcolor{blue}{m}}]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1} \colon |\textcolor{violet}{h}_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\} \quad m = m_\alpha = \alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|\textcolor{brown}{E}_{\textcolor{blue}{m}}] \qquad \qquad \qquad \sigma =: m + \hat{\sigma}$$

$$= \qquad \qquad \qquad H^m_N(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp\bigl(\beta H_N(m)\bigr)\, Q_N[\exp\left(\beta \textcolor{violet}{h}_{eff}(m) \cdot \hat{\sigma} + \beta H^m_N(\hat{\sigma})\right) | \textcolor{brown}{E}_{\textcolor{blue}{m}}]$$

$$\cong$$

$$\exp\bigl(\beta H_N(m)\bigr)\, Q_N[\exp\bigl(\beta H^m_N(\hat{\sigma})\bigr) | \textcolor{brown}{E}_{\textcolor{blue}{m}}]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|D_\alpha]$$

$$E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1} \colon |h_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\} \quad m = m_\alpha = \alpha u$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m] \qquad \qquad \qquad \sigma =: m + \hat{\sigma}$$

$$= \qquad \qquad \qquad \textcolor{red}{H}_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp\bigl(\beta H_N(m)\bigr)\, Q_N[\exp\left(\beta h_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)\,|E_m]$$

$$\cong$$

$$\exp\bigl(\beta H_N(m)\bigr)\, Q_N[\exp\bigl(\beta \textcolor{red}{H}_N^m(\hat{\sigma})\bigr)\,|E_m]$$

$$Q_N[\exp(\beta H_N(\sigma)) \mid D_\alpha]$$

$$E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1} : |h_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\} \quad m = m_\alpha = \alpha u$$

$$Q_N[\exp(\beta H_N(\sigma)) \mid E_m] \quad \sigma =: m + \hat{\sigma}$$

$$= \quad \quad \quad H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp(\beta H_N(m)) Q_N[\exp\left(\beta h_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})\right) \mid E_m]$$

$$\cong$$

$$\underbrace{\exp(\beta H_N(m)) Q_N[\exp(\beta \textcolor{red}{H}_N^m(\hat{\sigma})) \mid E_m]}_{\leq \exp\left(N \frac{\beta^2}{2} z_{\alpha^2} (1 - \alpha^2)\right)}$$

$$Q_N[\exp(\beta H_N(\sigma)) \mid D_\alpha]$$

$$E_m \coloneqq D_\alpha \cap \{\sigma \in S_{N-1} : |h_{eff}(m) \cdot (\sigma - m)| \leq N^{-1/3}\} \quad m = m_\alpha = \alpha u$$

$$Q_N[\exp(\beta H_N(\sigma)) \mid E_m] \quad \sigma =: m + \hat{\sigma}$$

$$= \quad H_N^m(\hat{\sigma}) \sim z_{\alpha^2}$$

$$\exp(\beta H_N(m)) Q_N[\exp(\beta h_{eff}(m) \cdot \hat{\sigma} + \beta H_N^m(\hat{\sigma})) \mid E_m]$$

$$\cong$$

$$\underbrace{\exp(\beta H_N(m)) Q_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m]}_{\leq \exp\left(N \frac{\beta^2}{2} z_{\alpha^2} (1 - \alpha^2)\right)}$$

$$Z_N = Z_N(D_0) + \sum_{\substack{\alpha \in A \setminus \{0\} \\ \lesssim \\ \exp\left(N \frac{\beta^2}{2} z(1)\right)}} Z_N(D_\alpha)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma)) | E_m] \cong \exp(\beta H_N(m)) \underbrace{Q_N[\exp(\beta H_N^m(\hat{\sigma})) | E_m]}_{\lesssim \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right)}$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(D_\alpha) \cong Q_N[\exp(\beta H_N(\sigma)) | D_\alpha] \times \exp(N\beta h\alpha) \times Q_N[D_\alpha]$$

$$Q_N[\exp(\beta H_N(\sigma)) | E_m] \cong \exp(\beta H_N(m)) Q_N[\exp(\beta H_N^m(\hat{\sigma})) | E_m]$$

$\lesssim \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right)$

$$\begin{array}{lcl} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(\textcolor{red}{E_m}) \cong Q_N[\exp(\beta H_N(\sigma)) \, | \textcolor{red}{E_m}] \times \exp(N\beta h\alpha) \times Q_N[\textcolor{red}{E_m}]$$

$$Q_N[\exp(\beta H_N(\sigma)) \, | \textcolor{red}{E_m}] \cong \exp\bigl(\beta H_N(m)\bigr) \, Q_N[\exp\bigl(\beta H_N^m(\hat{\sigma})\bigr) \, | \textcolor{red}{E_m}]$$

$$\begin{array}{lcl} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(E_m)\cong Q_N[\exp(\beta H_N(\sigma))\,|E_m]\times \exp(N\beta h\alpha)\times Q_N[E_m]$$

$$Q_N[\exp(\beta H_N(\sigma))\,|E_m]\cong \exp\bigl(\beta H_N(m)\bigr)\,Q_N[\exp\bigl(\beta H_N^m(\hat\sigma)\bigr)\,|E_m]$$

$$\begin{array}{lcl} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$\begin{array}{c} Z_N(E_m) \cong Q_N[\exp(\beta H_N(\sigma)) \, | E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m] \\ \cong \\ \exp\big(\beta H_N(m)\big) \, Q_N[\exp\big(\beta H_N^m(\hat\sigma)\big) \, | E_m] \end{array}$$

$$\begin{array}{lcl} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(\textcolor{blue}{E_m}) \cong Q_N[\exp(\beta H_N(\sigma)) \, | \textcolor{blue}{E_m}] \times \exp(N\beta h\alpha) \times Q_N[\textcolor{blue}{E_m}]$$

$$\begin{array}{lll} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(E_m)\cong Q_N[\exp(\beta H_N(\sigma))\,|E_m]\times \exp(N\beta h\alpha)\times Q_N[E_m]$$

$$\begin{aligned} Z_N &= Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha) \\ &\stackrel{\lesssim}{\leq} \exp\left(N \frac{\beta^2}{2} z(1)\right) \end{aligned}$$

$$\begin{aligned} Z_N(E_m) &\cong Q_N[\exp(\beta H_N(\sigma)) | E_m] \times \exp(N\beta h\alpha) \times Q_N[E_m] \\ &\cong \\ &\exp(\beta H_N(m)) Q_N[\exp(\beta H_N^m(\hat{\sigma})) | E_m] \end{aligned}$$

$$\begin{array}{lll} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(E_m)\cong \textcolor{red}{\exp\big(\beta H_N(m)\big)\times Q_N[\exp\big(\beta H_N^m(\hat\sigma)\big)\mid E_m]\times \exp(N\beta h\alpha)\times Q_N[E_m]}$$

$$\begin{array}{lll} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(E_m)\cong \exp\big(\beta H_N(m)\big)\times Q_N[\exp\big(\beta H_N^m(\hat\sigma)\big)\mid E_m]\times \exp(N\beta h\alpha)\times Q_N[E_m]$$

$$\begin{array}{lll} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(E_m)\cong \exp\bigl(\beta H_N(m)\bigr)\times \exp(N\beta h\alpha)\times Q_N[E_m]\times Q_N[\exp\bigl(\beta H_N^m(\hat\sigma)\bigr)\mid E_m]$$

$$\begin{aligned} Z_N &= Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha) \\ &\stackrel{\sim}{\leq} \exp\left(N \frac{\beta^2}{2} z(1)\right) \end{aligned}$$

$$\begin{aligned} Z_N(E_m) &\cong \exp(\beta H_N(m)) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N[\exp(\beta H_N^m(\hat{\sigma})) | E_m] \\ &\stackrel{\sim}{\leq} \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right) \end{aligned}$$

$$\begin{aligned} Z_N &= Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha) \\ &\stackrel{\sim}{\leq} \exp\left(N \frac{\beta^2}{2} z(1)\right) \end{aligned}$$

$$\begin{aligned} Z_N(E_m) &\cong \exp(\beta H_N(m)) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m] \\ &\stackrel{\sim}{\leq} \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right) \end{aligned}$$

$$Z_N = Z_N(D_0) + \sum_{\substack{\alpha \in A \setminus \{0\} \\ \lesssim \\ \exp\left(N\frac{\beta^2}{2}z(1)\right)}} Z_N(D_\alpha)$$

$$\begin{aligned} Z_N(E_m) &\cong \exp(\beta H_N(m)) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m] \\ &\qquad\qquad\qquad \cong \qquad\qquad\qquad \lesssim \\ &\qquad\qquad\qquad \exp\left(\frac{N}{2}\log(1-\alpha^2)\right) \exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right) \end{aligned}$$

$$Z_N = Z_N(D_0) + \sum_{\substack{\alpha \in A \setminus \{0\} \\ \lesssim \\ \exp\left(N\frac{\beta^2}{2}z(1)\right)}} Z_N(D_\alpha)$$

$$\begin{aligned} Z_N(E_m) &\cong \exp(\beta H_N(m)) \times \exp(N\beta h\alpha) \times Q_N[E_m] \times Q_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m] \\ &\quad \cong \qquad \cong \qquad \lesssim \\ &\quad \exp(\beta h(m \cdot u)) \exp\left(\frac{N}{2}\log(1 - \alpha^2)\right) \exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1 - \alpha^2)\right) \end{aligned}$$

$$\begin{aligned} Z_N &= Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha) \\ &\stackrel{\lesssim}{\leq} \exp\left(N \frac{\beta^2}{2} z(1)\right) \end{aligned}$$

$$\begin{aligned} Z_N(E_m) &\cong \exp(\beta H_N(m)) \times \exp(\beta h(m \cdot u)) \times Q_N[E_m] \times Q_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m] \\ &\quad \cong \stackrel{\lesssim}{\leq} \\ &\quad \exp\left(\frac{N}{2} \log(1 - \alpha^2)\right) \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right) \end{aligned}$$

$$\begin{array}{lcl} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{0\}} Z_N(D_\alpha) \\ & \lesssim & \\ & & \exp \left(N \frac{\beta^2}{2} z(1) \right) \end{array}$$

$$\begin{array}{lcl} Z_N(E_m) & \cong & \exp(\beta H_N(m) + \textcolor{blue}{h}(m \cdot u)) \times \textcolor{green}{Q}_N[E_m] \times \textcolor{red}{Q}_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m] \\ & & \cong \qquad \qquad \qquad \lesssim \\ & & \textcolor{green}{\exp\left(\frac{N}{2}\log(1-\alpha^2)\right)} \textcolor{red}{\exp\left(N\frac{\beta^2}{2}z_{\alpha^2}(1-\alpha^2)\right)} \end{array}$$

$$\begin{aligned} Z_N &= Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha) \\ &\stackrel{\lesssim}{\leq} \exp\left(N \frac{\beta^2}{2} z(1)\right) \end{aligned}$$

$$\begin{aligned} Z_N(E_m) &\cong \textcolor{blue}{\exp(\beta H_N(m) + h(m \cdot u))} \times \textcolor{green}{Q_N[E_m]} \times \textcolor{red}{Q_N[\exp(\beta H_N^m(\hat{\sigma})) \mid E_m]} \\ &\stackrel{\cong}{=} \textcolor{green}{\exp\left(\frac{N}{2} \log(1 - \alpha^2)\right)} \textcolor{red}{\exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right)} \end{aligned}$$

$$Z_N = Z_N(D_0) + \sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)$$

$$\lesssim \exp\left(N \frac{\beta^2}{2} z(1)\right)$$

$$Z_N(E_m) \cong \exp(\beta H_N(m) + h(m \cdot u)) \times Q_N[E_m] \times Q_N[\exp(\beta H_N^m(\hat{\sigma})) | E_m]$$

$$= \exp\left(\beta H_N^h(m)\right) \cong \exp\left(\frac{N}{2} \log(1 - \alpha^2)\right) \exp\left(N \frac{\beta^2}{2} z_{\alpha^2}(1 - \alpha^2)\right) \lesssim$$

$$\underbrace{\exp\left(\beta H_N^h(m) + \frac{N}{2} \log(1 - q) + N \frac{\beta^2}{2} z_q(1 - q)\right)}_{F_{TAP}(m)} \quad q := \frac{|m|^2}{N}$$

$$\begin{array}{lcl} Z_N & = & Z_N(D_0) \quad + \sum\limits_{\alpha \in A \backslash \{ 0 \}} Z_N(D_\alpha) \\ & \lesssim & \exp \left(N \dfrac{\beta^2}{2} z(1) \right) \end{array}$$

$$Z_N(E_m)\stackrel{\sim}{\leq}\exp(F_{TAP}(m))$$

$$F_{TAP}(m)\textcolor{red}{:=}\beta H_N^h(m)+\frac{N}{2}\log(1-q)+N\frac{\beta^2}{2}z_q(1-q)$$

$$q := \frac{|m|^2}{N}$$

TAP: Thouless-Anderson-Palmer '77

The corresponding free energy is not easily obtained from the Bethe method, and we therefore present it as a *fait accompli*, originally derived by diagram expansion :

$$\begin{aligned} F_{\text{MF}} = & - \sum_{(ij)} J_{ij} m_i m_j - \frac{1}{2} \beta \sum_{(ij)} J_{ij}^2 (1 - m_i^2)(1 - m_j^2) \\ & + \frac{1}{2} T \sum_i [(1 + m_i) \ln \frac{1}{2}(1 + m_i) + (1 - m_i) \ln \frac{1}{2}(1 - m_i)] \quad (13) \end{aligned}$$

Solution of 'Solvable model of a spin glass'

By D. J. THOULESS

Department of Mathematical Physics, University of Birmingham,
Birmingham, England

and P. W. ANDERSON†‡ and R. G. PALMER

Department of Physics, Princeton University,
Princeton, New Jersey 08540, U.S.A.‡

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both regions, they are far from complete analyses. We also encounter some 'coincidences' which require further investigation. Details of our solutions will be given elsewhere, and we attempt here only a general description of the methods.

Keep decomposing

$$\begin{aligned} Z_N &= \sum_m Z_N(E_m) \\ &\cdot \leq \exp(F_{TAP}(m) + o(N)) \end{aligned}$$

Theorem (B '21):

- H_N and mixed p -spin Hamiltonian
- Either
 - Spherical model (Q_N uniform on S_{N-1}), or
 - Ising model (Q_N uniform on $\{-1,1\}^N$)



$$F_N \leq \sup_m \frac{1}{N} F_{TAP}(m) + o(1)$$