Mixed p-spin FE when $h > 0$

$\Rightarrow$

TAP Free Energy (Thouless-Anderson-Palmer)

$$\text{Recall:}\left\{\begin{array}{l} \left.\begin{array}{c} z(x) = \sum_{p \geq 2} a_p \, x^p \\ h = 0 \quad \beta \in [0, \infty) \end{array}\right\} \Rightarrow F_N(\beta) \leq \frac{\beta^2}{2} z(1) + o(1) \\[3em] \left.\begin{array}{c} z(x) = \sum_{p \geq 2} a_p \, x^p \\ h = 0 \quad \beta \in [0, \beta_c] \end{array}\right\} \Rightarrow F_N(\beta) = \frac{\beta^2}{2} z(1) + o(1) \end{array}\right.$$

What about $\boldsymbol{h > 0}$?

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Model: spherical mixed p -spin

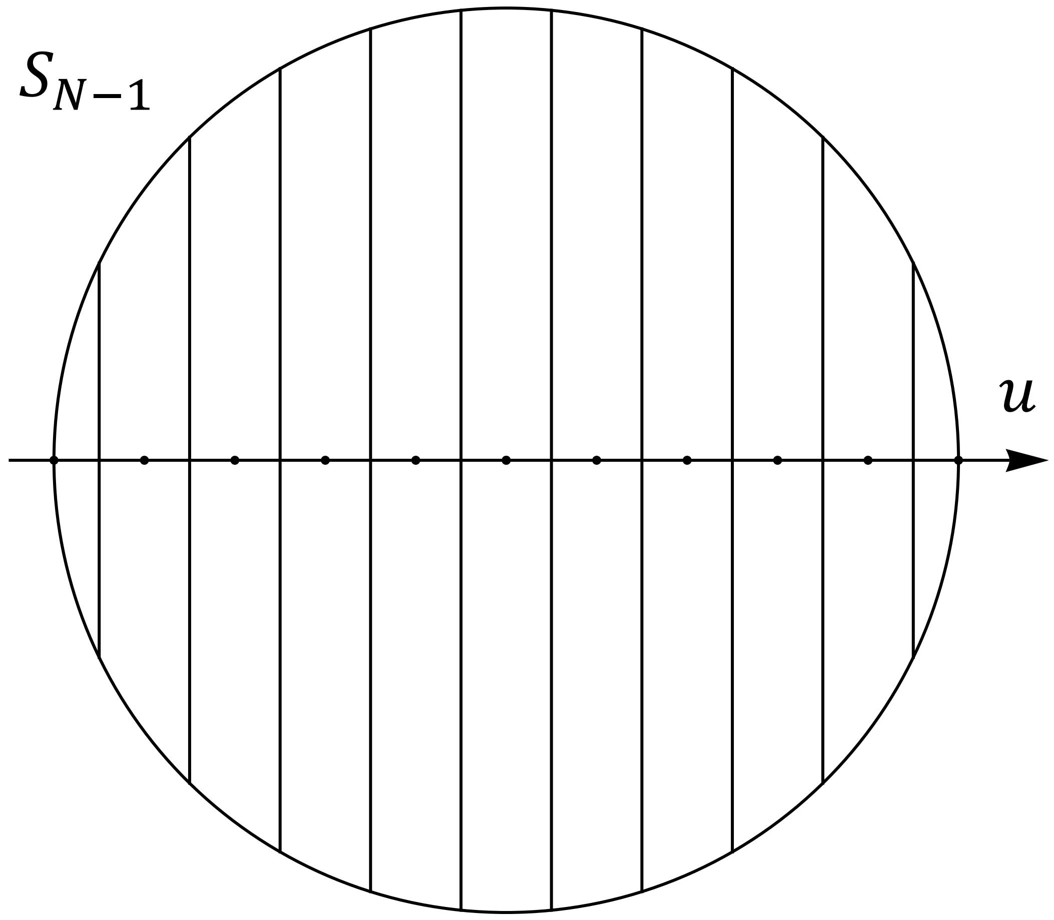
$$H_N(\sigma) \sim z(x) = \sum_{p\geq 2} a_p\, x^p$$

$$H_N^h(\sigma) = H_N(\sigma) + h(\sigma \cdot u) \qquad u \in \mathbb{R}^N, |u| = \sqrt{N}$$

$$\underbrace{F_N(\beta, h) = \frac{1}{N}\log Z_N(\beta, h) = \frac{1}{N}\log Q_N[\exp(\beta H_N^h(\sigma))]}_{\cdot\, = ???+o(1)}$$

C-W strategy: decompose in direction u

$$A := \left(N^{-1/3}\mathbb{Z}\right) \cap (-1,1)$$



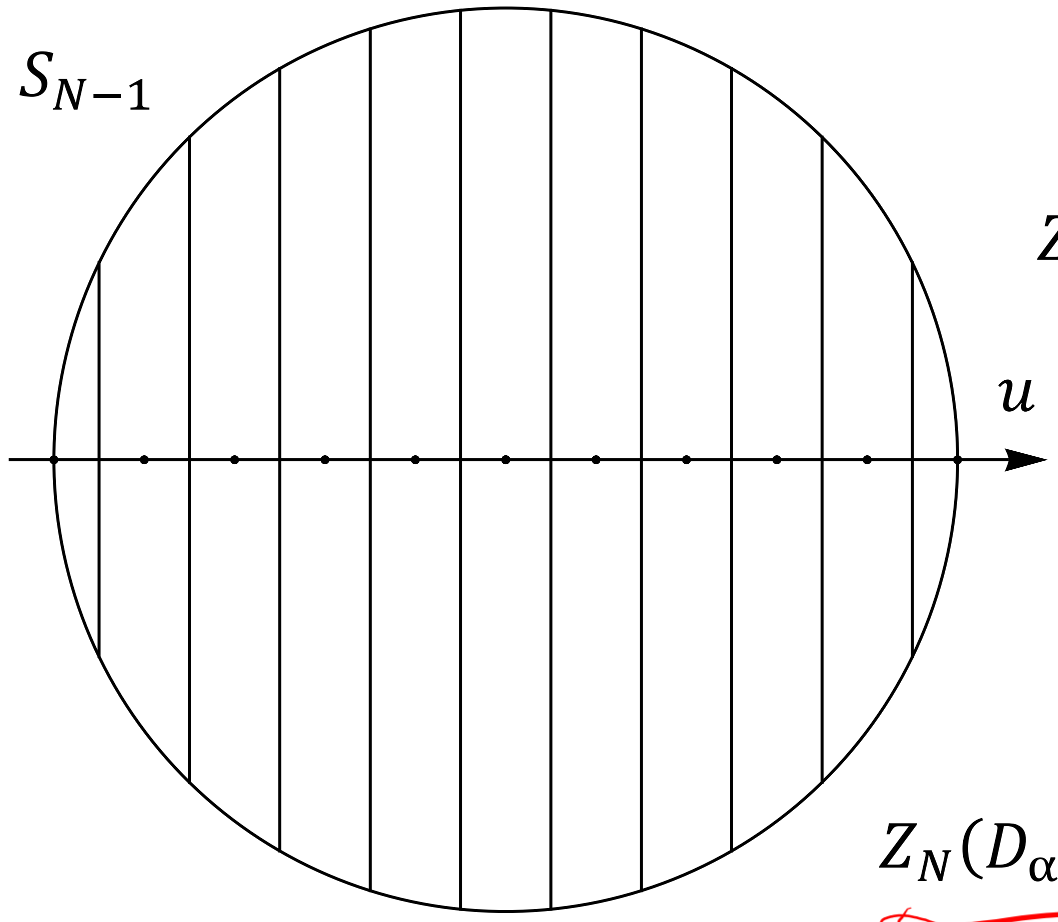
$$D_\alpha := \left\{\sigma \in S_{N-1} : \left|\frac{\sigma \cdot u}{N} - \alpha\right| \le N^{-1/3}\right\}$$

$$\Rightarrow \quad S_{N-1} = \bigcup_{\alpha \in A} D_\alpha$$

$$Z_N(D) \coloneqq Q_N\left[1_D \exp\left(\beta H_N^h(\sigma)\right)\right]$$

$$\Rightarrow \quad Z_N = Z_N(S_{N-1}) = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Ext. field term constant on D_α



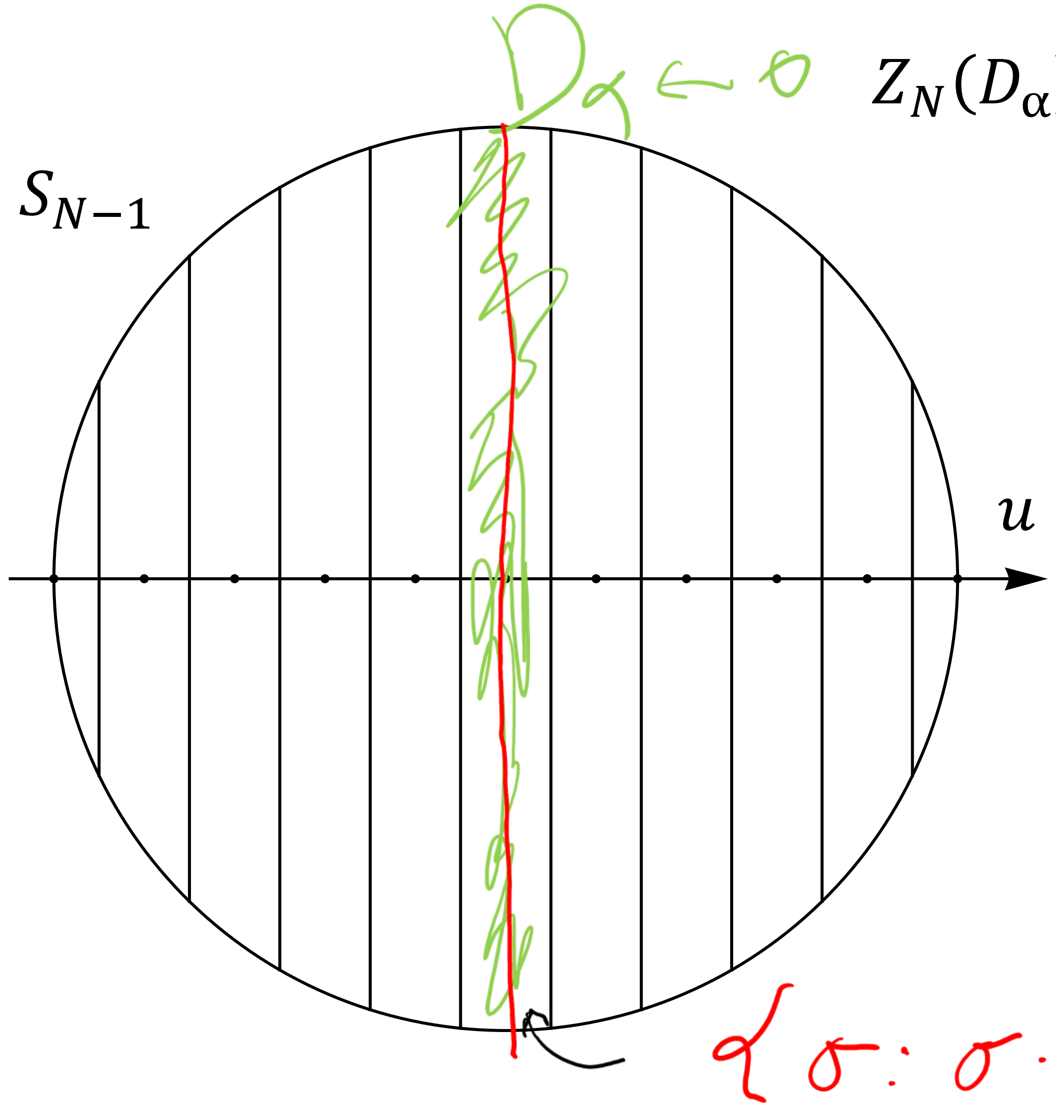
$$D_\alpha := \left\{\sigma \in S_{N-1} : \left|\frac{\sigma \cdot u}{N} - \alpha\right| \le N^{-1/3}\right\}$$

$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha} \exp\left(\beta H_N(\sigma) + \underbrace{\beta h(\sigma \cdot u)}_{\substack{= \\ N\beta h\alpha + o(N) \\ \text{on } D_\alpha}}\right)\right]$$

$$\Longrightarrow$$

$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha} \exp\left(\beta H_N(\sigma)\right)\right] \exp(N\beta h\alpha + o(N))$$

Estimating $Z_N(D_\alpha)$ for $\alpha = 0$



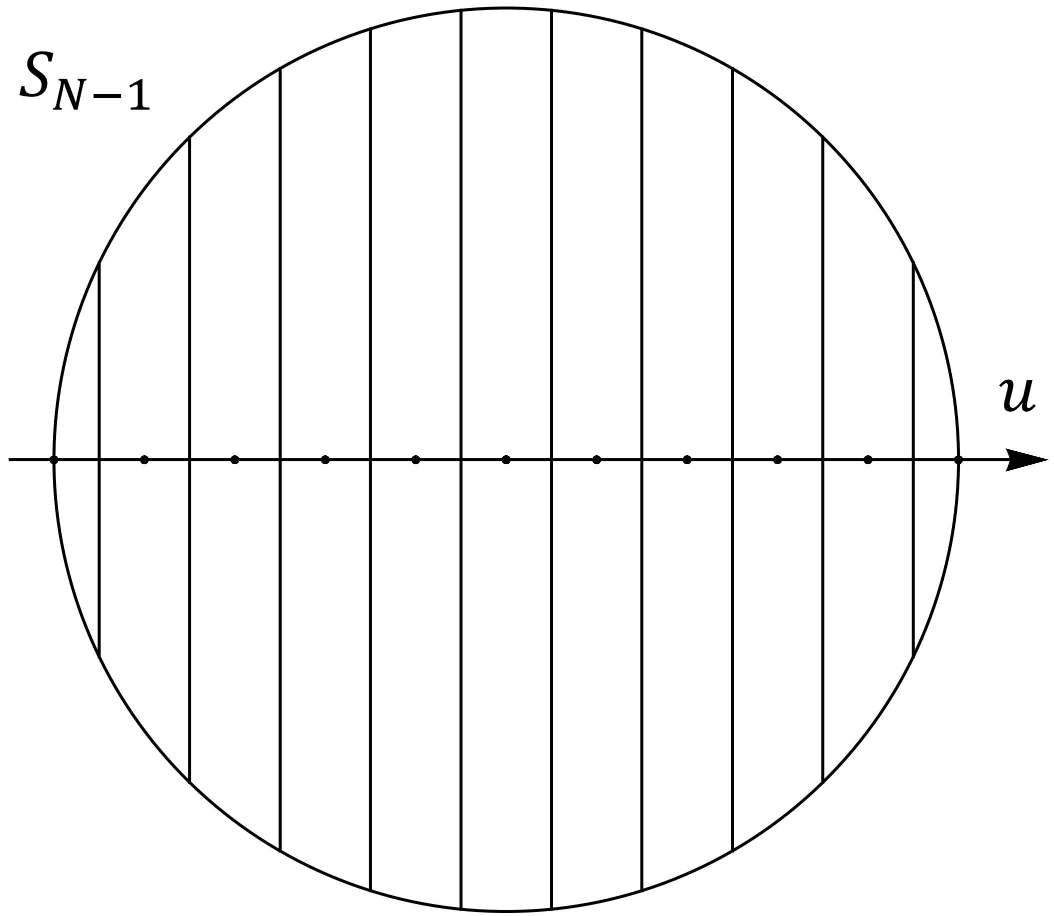
$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha}\exp\left(\beta H_N(\sigma)\right)\right]\exp(N\beta h\alpha + o(N))$$

For $\alpha = 0$:

$$Z_N(D_0) = \boxed{Q_N\left[1_{D_0}\exp\left(\beta H_N(\sigma)\right)\right]} e^{o(N)}$$

$$= \underbrace{Q_N[D_0]}_{=1+o(1)}\underbrace{Q_N\left[\exp\left(\beta H_N(\sigma)\right)|D_0\right]}_{\leq E[\;]^2 = \exp\left(N\frac{\beta^2}{2}\xi(1)\right)} e^{o(N)}$$

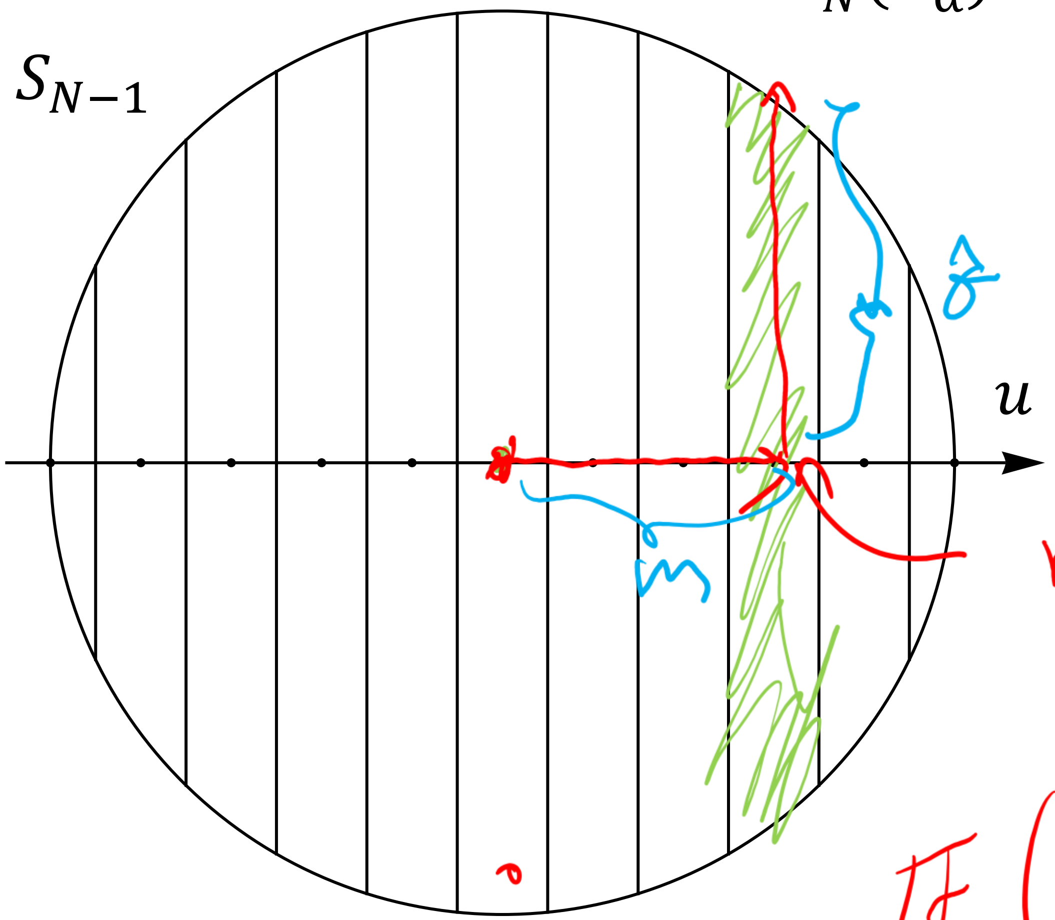
Consequence for decomposition



$$Z_N = Z_N(D_0) + \underline{\sum_{\alpha \in A \setminus \{0\}} Z_N(D_\alpha)}$$

$$Z_N(D_0) \leq \exp\left(N \frac{\beta^2}{2} z(1) + o(N)\right)$$

Estimating $Z_N(D_\alpha)$ for $\boldsymbol{\alpha} \neq 0$



$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha}\exp\left(\beta H_N(\sigma)\right)\right]\exp(N\beta h\alpha)e^{o(N)}$$

$$= Q_N[D_\alpha]Q_N\left[\exp\left(\beta H_N(\sigma)\right)|D_\alpha\right]\exp(N\beta h\alpha)e^{o}$$

$$= \exp\left(N\frac{\beta^2}{2}Z(1)\right)$$

$m = \sigma u$

$m + \hat{\sigma}$ $\quad$ $m + \hat{\tau}$

$(m + \hat{\sigma})\cdot(m + \hat{\tau}) = |m|^2 > 0$

$$\mathbb{E}\left(H_N(\sigma)H_N(\tau)\right) \approx 0$$

$Z\left(\frac{\sigma\cdot\tau}{N}\right)$

Recentering Hamiltonian around m

P_α

$$H_N(m+\hat{\sigma}) = H_N(m) + \nabla H_N(m)\cdot\hat{\sigma} + \dots$$

$$H_N(m+\hat{\sigma}) =: H_N(m) + \nabla H_N(m)\cdot\hat{\sigma} + H_N^m(\sigma)$$

$$Q_N\left[\exp\left(\beta H_N(\sigma)\right) \middle| D_\alpha\right] = Q_N\left[\exp\left(\beta H_N(m) + \beta\nabla H_N(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right) \middle| D_\alpha\right]$$

$$\underbrace{H_N(m) \qquad\qquad \left(\nabla H_N(m)\cdot\hat{\sigma}\right)_{\hat{\sigma}:\hat{\sigma}\cdot m=0} \qquad\qquad \left(H_N^m(\hat{\sigma})\right)_{\hat{\sigma}:\hat{\sigma}\cdot m=0}}_{\text{Independent Gaussian processes!}}$$