Mixed p-spin FE when $h > 0$

TAP Free Energy
(Thouless-Anderson-Palmer)

Recall:

$$z(x) = \sum_{p \geq 2} a_p \, x^p$$

$h = 0 \quad \beta \in [0, \infty)$

$$\Rightarrow \quad F_N(\beta) \leq \frac{\beta^2}{2} z(1) + o(1)$$

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Model: spherical mixed p -spin

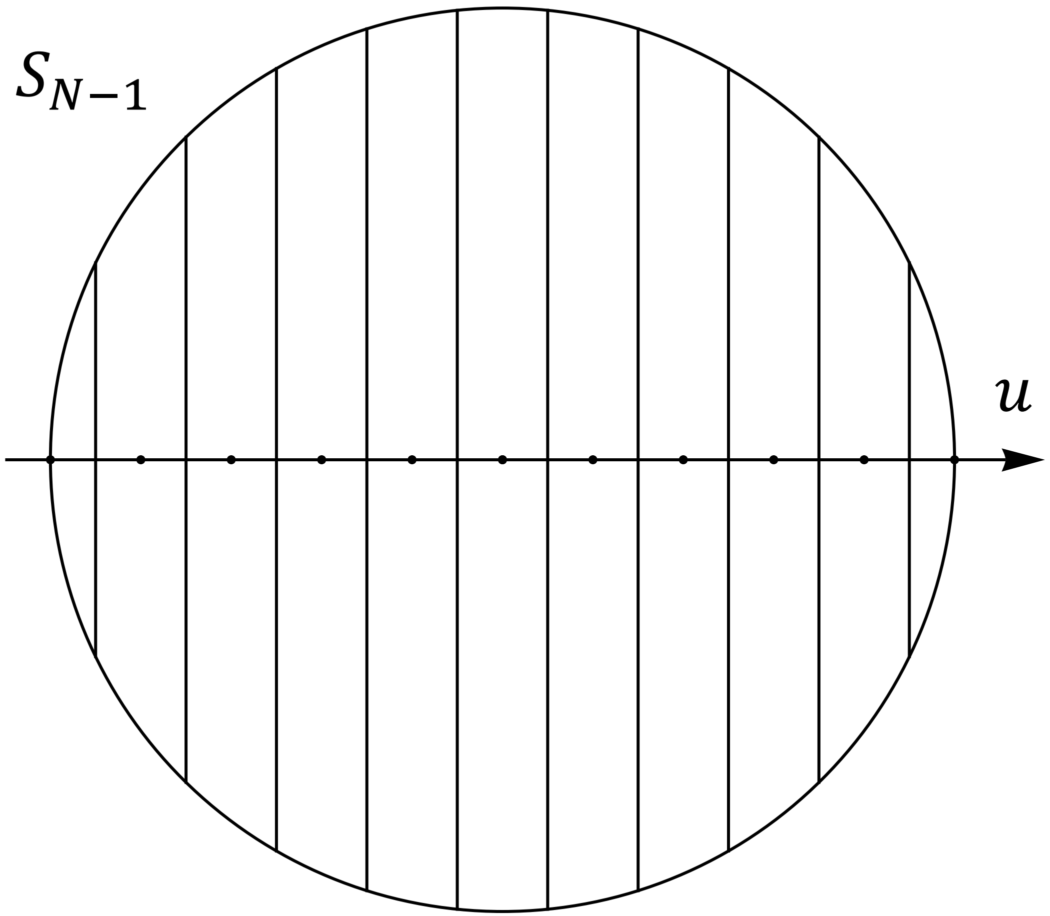
$$H_N(\sigma) \sim z(x) = \sum_{p\geq 2} a_p\, x^p$$

$$H_N^h(\sigma) = H_N(\sigma) + h(\sigma\cdot u) \qquad u \in \mathbb{R}^N, |u| = \sqrt{N}$$

$$\underbrace{F_N(\beta,h) = \frac{1}{N}\log Z_N(\beta,h) = \frac{1}{N}\log Q_N[\exp(\beta H_N^h(\sigma))]}_{\cdot\, =???+o(1)}$$

C-W strategy: decompose in direction u

$A := \left(N^{-1/3}\mathbb{Z}\right) \cap (-1,1)$



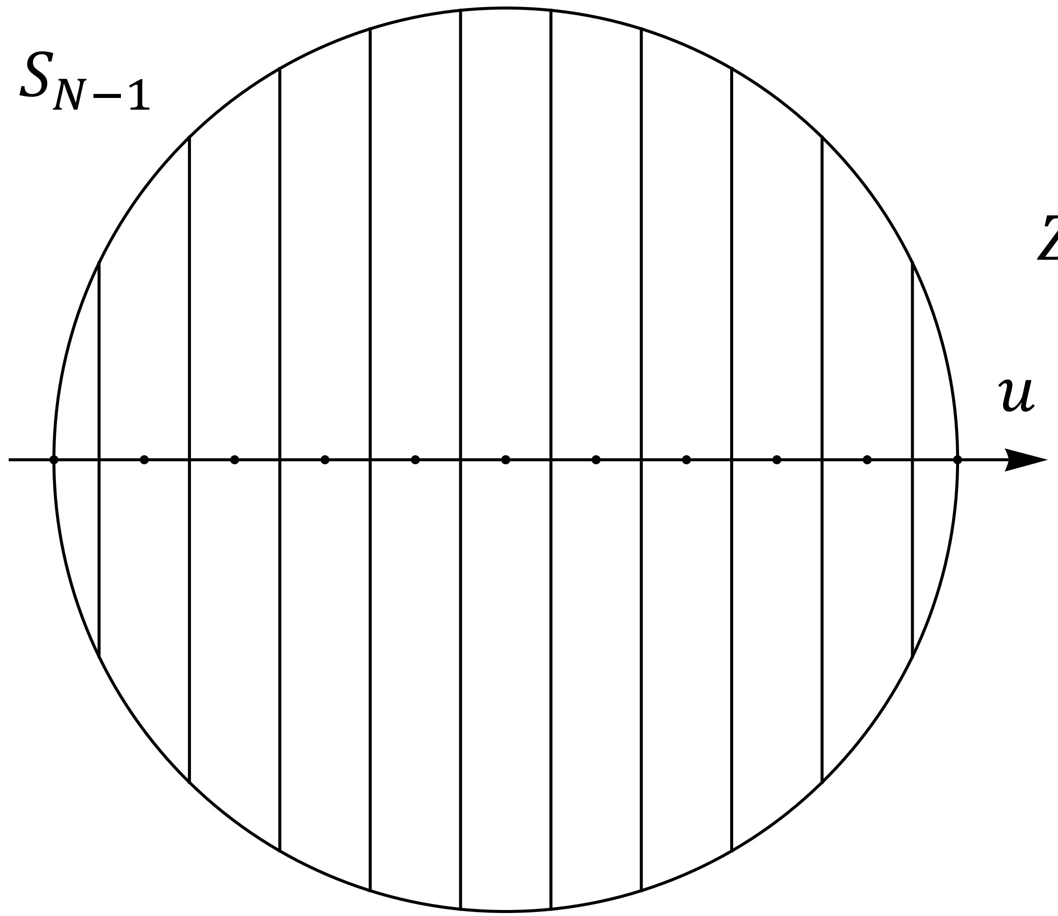
$$D_\alpha := \left\{\sigma \in S_{N-1} : \left|\frac{\sigma \cdot u}{N} - \alpha\right| \le N^{-1/3}\right\}$$

$$\Longrightarrow \quad S_{N-1} = \bigcup_{\alpha \in A} D_\alpha$$

$$Z_N(D) \coloneqq Q_N\left[1_D \exp\left(\beta H_N^h(\sigma)\right)\right]$$

$$\Longrightarrow \quad Z_N = Z_N(S_{N-1}) = \sum_{\alpha \in A} Z_N(D_\alpha)$$

Ext. field term constant on D_α

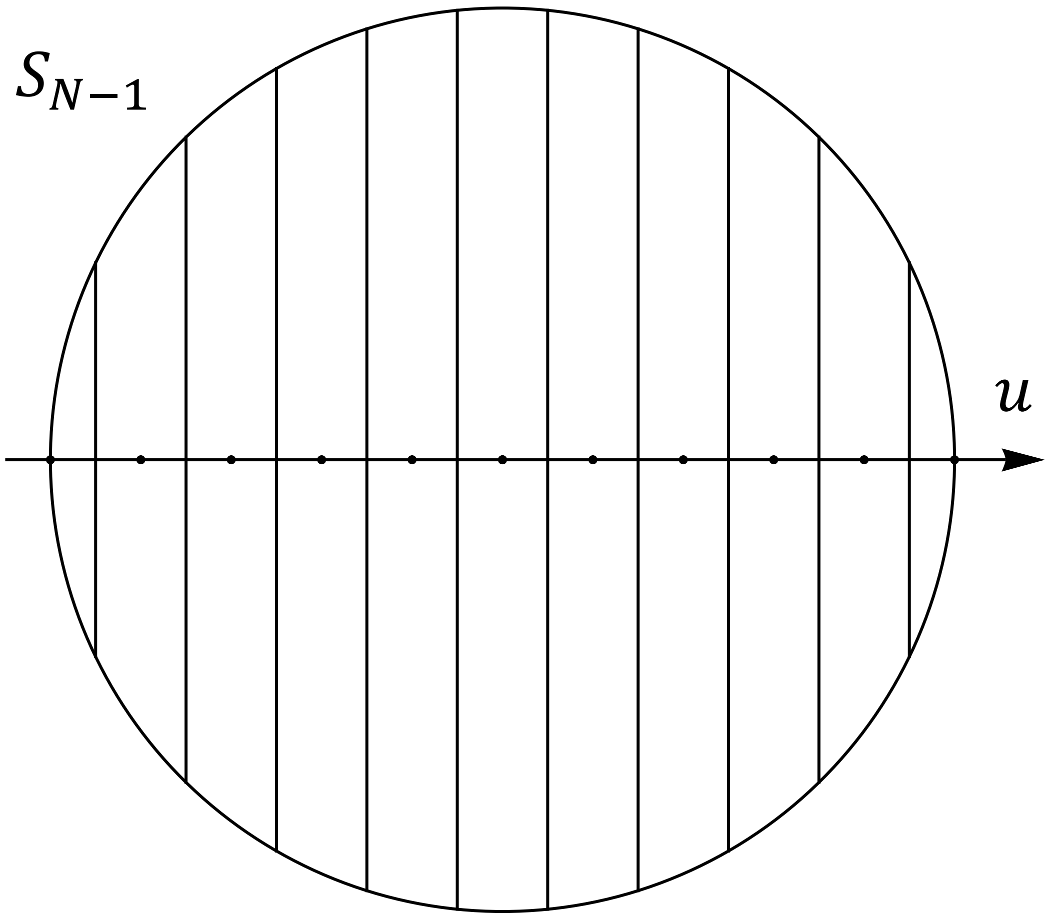


$$D_\alpha := \left\{ \sigma \in S_{N-1} : \left| \frac{\sigma \cdot u}{N} - \alpha \right| \le N^{-1/3} \right\}$$

$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha} \exp\left(\beta H_N(\sigma) + \underbrace{\beta h(\sigma \cdot u)}_{\substack{= \\ N\beta h\alpha + o(N) \\ \text{on } D_\alpha}} \right) \right]$$

$\Rightarrow$

Estimating $Z_N(D_\alpha)$ for $\alpha = 0$

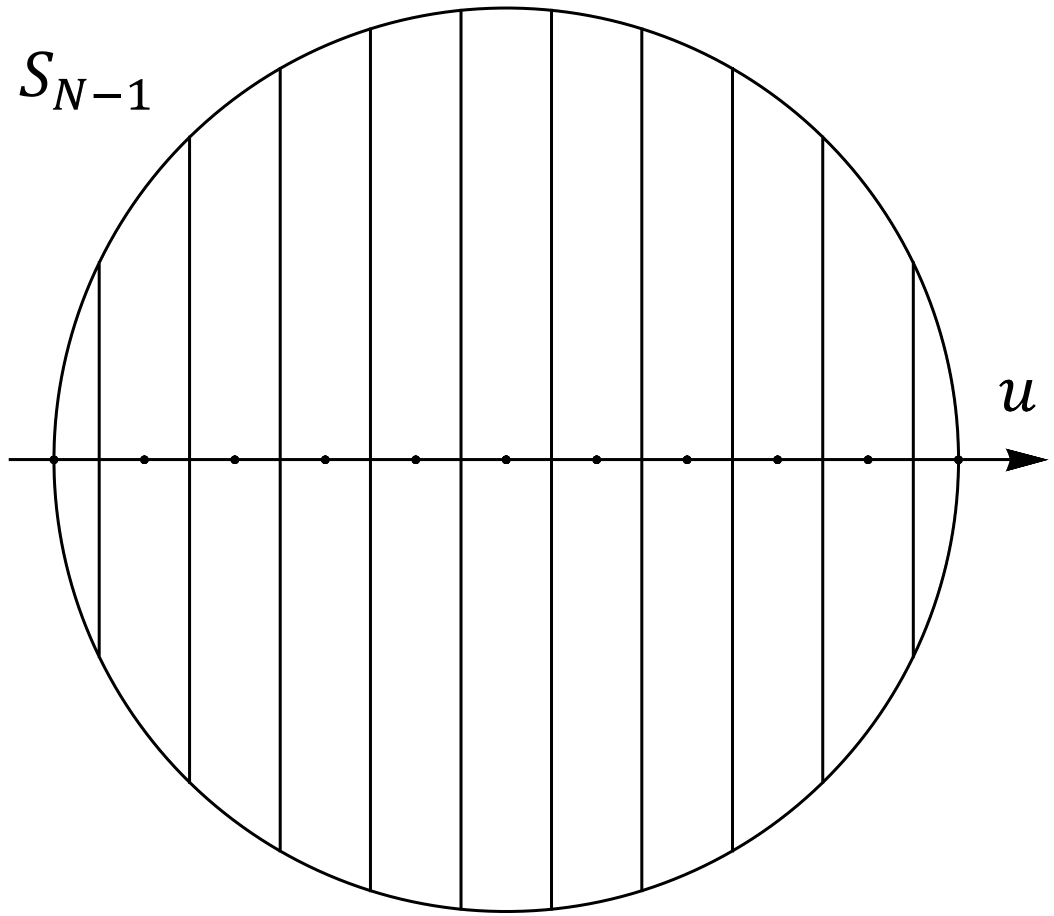


$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha} \exp\left(\beta H_N(\sigma)\right)\right] \exp(N\beta h\alpha + o(N))$$

For $\alpha = 0$:

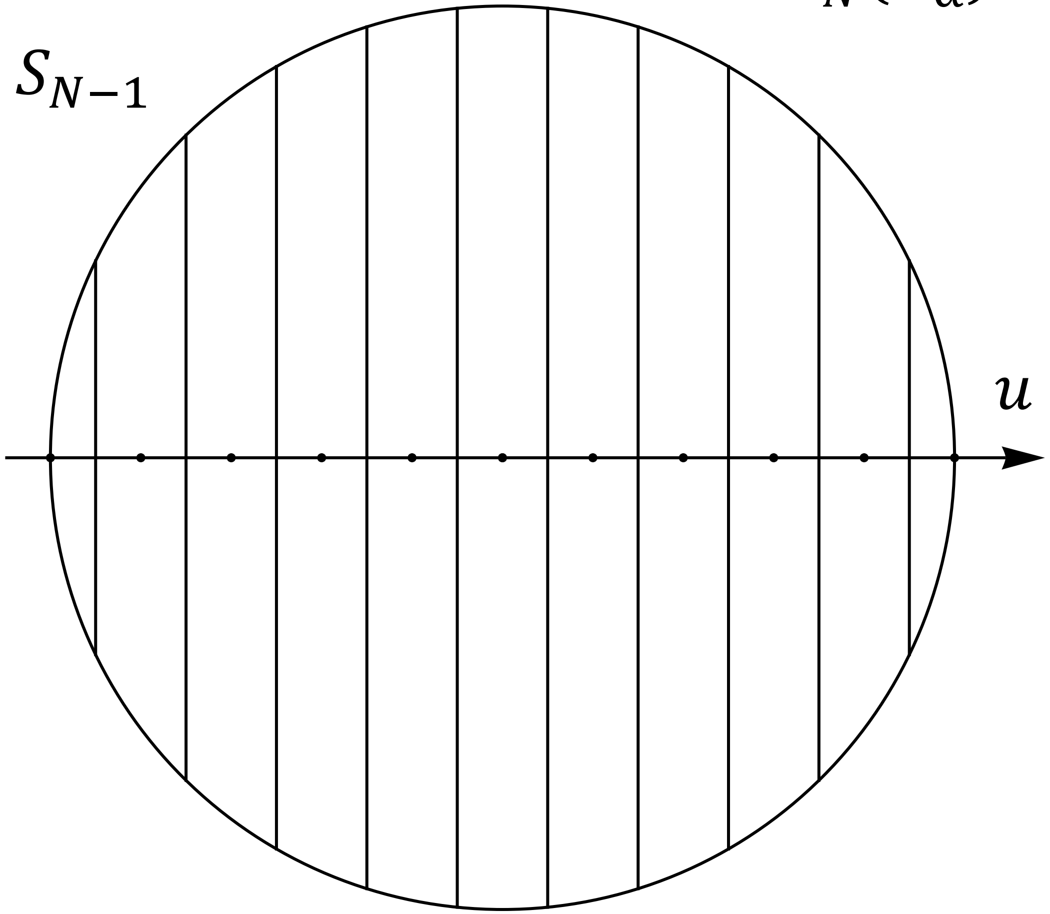
$$\begin{aligned} Z_N(D_0) &= Q_N\left[1_{D_0} \exp\left(\beta H_N(\sigma)\right)\right] e^{o(N)} \\ &= Q_N[D_0] Q_N\left[\exp\left(\beta H_N(\sigma)\right) | D_0\right] e^{o(N)} \end{aligned}$$

Consequence for decomposition



$$Z_N = \underbrace{Z_N(D_0)}_{\leq \exp\left(N\frac{\beta^2}{2}z(1)+o(N)\right)} + \sum_{\alpha\in A\setminus\{0\}} Z_N(D_\alpha)$$

Estimating $Z_N(D_\alpha)$ for $\alpha \neq 0$



$$Z_N(D_\alpha) = Q_N\left[1_{D_\alpha} \exp\left(\beta H_N(\sigma)\right)\right] \exp(N\beta h\alpha) e^{o(N)}$$
$$= Q_N[D_\alpha] Q_N\left[\exp\left(\beta H_N(\sigma)\right) | D_\alpha\right] \exp(N\beta h\alpha) e^{o}$$

Recentering Hamiltonian around m

$$H_N(m+\hat{\sigma}) = H_N(m) + \nabla H_N(m)\cdot\hat{\sigma} + \dots$$

$$H_N(m+\hat{\sigma}) =: H_N(m) + \nabla H_N(m)\cdot\hat{\sigma} + H_N^m(\sigma)$$

$$Q_N\left[\exp\left(\beta H_N\,(\sigma)\right)|D_\alpha\right] = Q_N\left[\exp\left(\beta H_N(m) + \beta\nabla H_N(m)\cdot\hat{\sigma} + \beta H_N^m(\hat{\sigma})\right)|D_\alpha\right]$$

$$\underbrace{H_N(m) \qquad (\nabla H_N(m)\cdot\hat{\sigma})_{\hat{\sigma}:\hat{\sigma}\cdot m=0} \qquad \left(H_N^m(\hat{\sigma})\right)_{\hat{\sigma}:\hat{\sigma}\cdot m=0}}_{\text{Independent Gaussian processes!}}$$